The BCS Critical Temperature at High Density

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Abstract

In this paper, we investigate the BCS cricital temperature, T_c, at high densities in spatial dimensions $d = 2$ and $d = 3$. We find that the behavior of T_c for high densities strongly depends on the behavior of the interaction potential V near the Fermi-surface and provide asymptotic formulas for the critical temperature in this limit. Our results include a rigorous confirmation for the behavior of T_c at high densities proposed by Langmann *et al.* in [14], from which they concluded the *ubiquity of superconducting domes in BCS theory*, which were observed, e.g., in doped band insulators or magic-angle graphene.

1 Introduction

The BCS gap equation [2]

$$\Delta(p) = -\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{V}(p-q) \frac{\Delta(q)}{E(q)} \tanh\left(\frac{E(q)}{2T}\right) \mathrm{d}q\,, \tag{1}$$

with $E(p) = \sqrt{(p^2-\mu)^2 + |\Delta(p)|^2}$, has an important role in physics since its introduction. The function Δ is interpreted as the order parameter describing paired Fermions (Cooper pairs) interacting via the local pair potential $2V$, where $\hat{V}(p) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} V(x) \mathrm{e}^{-\mathrm{i}p\cdot x} \mathrm{d}x$ denotes its Fourier transform. The parameters T and μ are the temperature and the chemical potential, respectively. The chemical potential μ might be interpreted as the density of Fermions.

In this paper we are interested in the critical temperature for the existence of non-trivial solutions of the BCS gap equation (1) in the high-density limit for dimensions $d = 2$ and $d = 3$. There are already rigorous results in the low-density [9] and weak coupling regime [5, 8]. Investigating the high-density limit of the critical temperature is especially relevant for explaining superconducting domes [12, 4, 19, 18, 20, 3], i.e. a non-monotonic $T_c(\mu)$ exhibiting a maximum value at finite μ and going to zero for large

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μ. In a recent paper [14], Langmann *et al.* claim the *ubiquity of superconducting domes in BCS theory* with finite-range potentials, but only for pure s-wave superconductivity (i.e. angular momentum $\ell = 0$, see Corollary 1). Their result disproves the conventional wisdom, that the presence of a superconducting dome necessarily indicates some kind of exotic superconductivity, e.g. resulting from competing orders. Our results are in particular relevant for doped band insulators [20] and magic-angle graphene [3], where no competing orders occur, and thus a more conventional explanation is necessary. The observation of superconducting domes in magic-angle graphene is the main motivation for studying the case $d = 2$.

There is a simple physical picture arising from an interplay of length scales, that explains the ubiquity of superconducting domes (see [14]). If the effective range ξ of the interaction is much smaller than the mean interparticle distance $\mu^{-1/2}$, i.e. $\xi \ll \mu^{-1/2}$, we can increase T_c by increasing μ as predicted by standard BCS theory [2] and rigorously justified in [9]. Whereas, at high densities, i.e. $\xi \gg \mu^{-1/2}$, the pairing of the electrons near the Fermi surface becomes weaker with increasing μ due to the decay of the interaction in Fourier space, suppressing T_c towards zero. Therefore, at intermediate densities, where $\xi \sim \mu^{-1/2}$, a superconducting dome arises. This simple argument is reflected in our results by the presence of the operator $\mathcal{V}_\mu^{(d)}$, defined in (2), acting on functions on the (rescaled) Fermi surface.

Our results are threefold: first, we confirm the claims from Langmann *et al.* [14] on the critical temperature at high densitites for s-wave superconductivity (to lowest order) by proving a more general result for radially symmetric interaction potentials V in $d = 3$ (Theorem 1 and Corollary 1); second, we provide upper and lower bounds for general non-sperically symmetric interaction potentials, partially based on a method used by Gontier *et al.* in [6], again for $d = 3$ (Theorem 2); third, we derive the asymptotic behavior of T_c at high densities in full generality for $d = 2$ (Theorem 3).

2 Main Results

2.1 Preliminaries

It was proven in [7] (see also [11] for a more recent review) that the critical temperature for the existence of non-trivial solutions of the BCS gap equation (1) can be characterized as follows.

Definition 1 (Critical Temperature)**.** Let $\mu > 0$, $d = 2, 3$ and

$$V \in \begin{cases} L^{1+\varepsilon}(\mathbb{R}^2) & \text{if } d = 2 \\ L^{3/2}(\mathbb{R}^3) & \text{if } d = 3 \end{cases}$$

be real-valued. Let $K_{T,\mu}$ denote the multiplication operator in momentum space

$$K_{T,\mu}(p) = \frac{|p^2 - \mu|}{\tanh\left(\frac{|p^2-\mu|}{2T}\right)}.$$

The critical temperature for the BCS gap equation is given by

$$\boxed{T_c = \inf \{T > 0 \mid K_{T,\mu}(p) + V(x) \geq 0\}}\,.$$

One might think of the operator $K_{T,\mu}(p)+V(x)$ as the Hessian in the BCS functional of superconductivity (see [11]), where the positivity is related to the "stability" of a superconducting state, which is related to the existence of a non-trivial solution of (1). Note, that by the Sobolev inequality (Theorem 8.3 and 8.5 in [17]) T_c is well defined for both cases $d = 2, 3$, as the continuous spectrum of $K_{T,\mu}$ starts at $2T$ for any μ and $K_{T,\mu} \sim p^2$ for large $|p|$.

Moreover, note that $K_{T,\mu}$ vanishes on the codim-1 submanifold $\{\, p^2 = \mu \,\}$, where its resolvent diverges. Thus, similarly to the weak coupling situation [5] and as pointed out by Laptev, Safronov and Weidl, [15] (see also [10]), the spectrum of $K_{T,\mu} + V$ is mainly determined by the behavior of V near $\{\, p^2 = \mu \,\}$, i.e. the Fermi sphere. More precisely, as empasized in the introduction, a crucial role for the investigation of T_c in the high-density limit is played by the (rescaled) operator $\mathcal{V}_\mu^{(d)} : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ where

$$\left(\mathcal{V}_\mu^{(d)} u\right)(p) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{S}^{d-1}} \hat{V}(\sqrt{\mu}(p-q))u(q)\,\mathrm{d}\omega(q)\,. \tag{2}$$

The evaluation of $\hat{V}$ on a codim-1 submanifold is well defined by the Riemann-Lebesgue Lemma for $V \in L^1(\mathbb{R}^d)$. The lowest eigenvalue of $\mathcal{V}_\mu^{(d)}$, which we denote by

$$e_\mu^{(d)} = \operatorname{infspec} \mathcal{V}_\mu^{(d)}$$

will be of particular importance. Note, that $\mathcal{V}_\mu^{(d)}$ is a compact operator (so $e_\mu^{(d)} \leq 0$), which is in fact trace class with

$$\operatorname{tr}(\mathcal{V}_\mu^{(d)}) = \frac{\left|\mathbb{S}^{d-1}\right|}{(2\pi)^d} \int_{\mathbb{R}^d} V(x)\mathrm{d}x\,.$$

The case $e_\mu^{(d)} < 0$ which will be important for our main theorems as it corresponds to an attractive interaction between (some) electrons on the Fermi sphere. Since $\mathcal{V}_\mu^{(d)}$ is trace class, a sufficient condition for $e_\mu^{(d)} < 0$ is that the trace of $\mathcal{V}_\mu^{(d)}$ is negative, i.e. $\int V < 0$. Moreover, by considering a trial function that is concentrated on two small sets on the rescaled Fermi sphere $\mathbb{S}^{d-1}$ separated by a distance $|p| < 2$, one can easily see that $e_\mu^{(d)} < 0$ if $|\hat{V}(p)| > \hat{V}(0)$ for some $|p| < 2\sqrt{\mu}$.

In the special case of radial potentials V depending only on $|x|$, the spectrum of $\mathcal{V}_\mu^{(d)}$ can be determined more explicitly (see, e.g., [5]). If V is radially symmetric, the eigenfunctions of $\mathcal{V}_\mu^{(d)}$ are spherical harmonics or circular harmonics for $d = 2, 3$, respectively. The corresponding eigenvalues are given by

$$\frac{\left|\mathbb{S}^{d-1}\right|}{(2\pi)^d} \int_{\mathbb{R}^d} V(x) \left(f_\ell^{(d)}(\sqrt{\mu}|x|)\right)^2 \mathrm{d}x\,,$$

where $f_\ell^{(2)} = J_\ell$ (Bessel function of the first kind) and $f_\ell^{(3)} = j_\ell = \sqrt{\frac{\pi}{2x}} J_{\ell+\frac{1}{2}}$ (spherical Bessel function) and thus

$$e_\mu^{(d)} = \frac{\left|\mathbb{S}^{d-1}\right|}{(2\pi)^d} \inf_{\ell\in\mathbb{N}_0} \int_{\mathbb{R}^d} V(x) \left(f_\ell^{(d)}(\sqrt{\mu}|x|)\right)^2 \mathrm{d}x\,.$$

Note that in this case we can bound

$$|e_\mu^{(d)}| \leq \frac{\left|\mathbb{S}^{d-1}\right|}{(2\pi)^d} \left\| \frac{V}{|\cdot|^{d-2+\alpha}} \right\|_1 \sup_{\ell\in\mathbb{N}_0} \left\| |\cdot|^{\frac{d-2+\alpha}{2}} f_\ell^{(d)} \right\|_\infty \frac{1}{\mu^{\frac{d-2+\alpha}{2}}} \tag{3}$$

by Hölder to get a simple estimate on the behavior of $e_\mu^{(d)}$ after supposing a certain integrability condition on V. The term involving $f_\ell^{(d)}$ is finite as long as $\alpha \leq 2/3$, which was shown in [13] (see also the proof of Theorem 1). If additionally $\hat{V} \leq 0$, the minimal eigenvalue is attained for the constant eigenfunction (i.e. the circular resp. spherical harmonic with $\ell = 0$) by the Perron-Frobenius Theorem and we thus have the more concrete expression

$$e_\mu^{(d)} = \frac{\left|\mathbb{S}^{d-1}\right|}{(2\pi)^d} \int_{\mathbb{R}^d} V(x) \left(f_0^{(d)}(\sqrt{\mu}|x|)\right)^2 \mathrm{d}x\,. \tag{4}$$

Here, the same bound holds as in (3) but the threshold of $\alpha \leq 2/3$ can be pushed to $\alpha \leq 1$ as there is no $\sup_{\ell\in\mathbb{N}_0}$ involved.

We emphasized above, that $e_\mu^{(d)}$ will play a prominent role in the following subsection. As it strongly depends on the concrete form of the interaction, we cannot give more explicit formulas, but the bound obtained in (3) and the special case of s-wave superconductivity (4) allows us to get an idea about the effective role of $e_\mu^{(d)}$ in the asymptotic formulas given below.

2.2 Results

As mentioned in the introduction, our results are threefold: First, we show the asymptotic formula for $d = 3$ and radial potentials, including the rigorous confirmation of the result from Langmann *et al.* [14] (Theorem 1 and Corollary 1). Afterwards, we provide upper and lower bounds in $d = 3$ for potentials which are no longer radially symmetric (Theorem 2). Finally, we give the corresponding asymptotic formula in full generality for $d = 2$ (Theorem 3).

Theorem 1. *Let $d = 3$. Let $V \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$ be real-valued and radially symmetric. Assume that there exists $\mu_0 > 0$ such that for all $\mu \geq \mu_0$ we have $e_\mu^{(3)} < 0$. Then $T_c(\mu) > 0$ for all sufficiently large μ and*

$$\lim_{\mu\to\infty} \sqrt{\mu}\, e_\mu^{(3)} \ln\left(\frac{\mu}{T_c}\right) = -1. \tag{5}$$

Or in other words, we have the asymptotic behavior

$$\boxed{T_c \sim \mu\, \mathrm{e}^{1/(\sqrt{\mu} e_\mu^{(3)})}}$$

in the limit of large μ. Note, that the right hand side is the same formula as in the weak-coupling case [5, 8] but with $\lambda = 1$. As a simple corollary, we rigorously confirm the result from [14].

Corollary 1. *Let V be as in Theorem 1 and assume additionally that $\hat{V} \leq 0$. Note, that this immediately implies that $e_\mu^{(3)} < 0$ for all μ. Then, using the notation from [14], we have*

$$\sqrt{\mu} e_\mu^{(3)} = \frac{\sqrt{\mu}}{2\pi^2} \int_{\mathbb{R}^d} V(x) \frac{\sin^2(\sqrt{\mu}|x|)}{\mu |x|^2} \mathrm{d}x = \frac{1}{4\pi^2} \frac{f_{-2V}(4\mu)}{4\sqrt{\mu}} =: -\lambda$$

and thereby confirm the validity of eq. (6) from [14] in the high-density limit for leading order, i.e.

$$T_c \sim \mu\, \mathrm{e}^{-1/\lambda}\,.$$

Proof. The first equality follows by (4) since $j_0(x) = \frac{\sin(x)}{x}$. The second equality is a simple computation using Fubini. □

For more general interactions, which are not neccesarily radially symmetric, we obtain upper and lower bounds on the critical temperature as our second result for $d = 3$. The upper bound uses an estimate from [6], which was originally derived for studies on a lower bound on the Hartree-Fock energy of the electron gas. This proof will thus be somewhat different from the proofs of the other statements.

Theorem 2. *Let $d = 3$.*

(a) Let V be a real-valued measureable function with $V|\cdot| \in L^\infty(\mathbb{R}^3)$. Then

$$T_c \lesssim \mu \exp\left(-\sqrt{\frac{\pi}{2\|V|\cdot|\|_\infty}} \mu^{1/4} \right)\,. \tag{6}$$

(b) Let $V \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$ be real-valued but <u>not</u> neccesarily radially symmetric. Assume that there exists $\mu_0 > 0$ such that for all $\mu \geq \mu_0$ we have $e_\mu^{(3)} < 0$. Then

$$T_c \gtrsim \mu \exp\left(\frac{\ln(\mu)}{|o(1)| \sqrt{\mu} e_\mu^{(3)}(V)} \right)\,, \tag{7}$$

where $o(1)$ vanishes as $\mu \to \infty$.

Part (a) shows that the critical temperature vanishes for a very large class of interaction potentials in the limit $\mu \to \infty$. Part (b) provides a lower bound that differs only a little from the correct asymptotics for radially symmetric potentials in Theorem 1 and might hence serve as as starting point for proving the correct asymptotics for more general potentials.

As a third result, we provide the high-density asymptotics of the critical temperature in $d = 2$ for general interaction potentials without assuming radial symmetry.

Theorem 3. *Let $d = 2$. Let $V \in L^1(\mathbb{R}^2) \cap L^{1+\varepsilon}(\mathbb{R}^2)$ be real-valued. Assume that there exists $\mu_0 > 0$ such that for all $\mu \geq \mu_0$ we have $e_\mu^{(2)}(V) < 0$. Then $T_c(\mu) > 0$ for all sufficiently large μ and*

$$\lim_{\mu\to\infty} e_\mu^{(2)}(V) \ln\left(\frac{\mu}{T_c}\right) = -1 \,. \tag{8}$$

This means that

$$\boxed{T_c \sim \mu\, \mathrm{e}^{1/e_\mu^{(2)}}}$$

in analogy to the asymptotics obtained in Theorem 1 for $d = 3$.

2.3 Superconducting domes

We conclude this section with a short discussion about superconducting domes. Our results show for a very general class of interaction potentials that the critical temperature in BCS theory vanishes in the limit $\mu \to \infty$. The heuristic reasoning that the decay of the interaction in Fourier space dominates the behavior of T_c in the high-density limit is indeed reflected by the presence of $e_\mu^{(d)}$ in the asymptotic formulas. The existence of a maximal critical temperature at some intermediate density (*superconducting dome*), can be obtained by combining the decay of T_c in the high-density limit from our main Theorems to the decay of T_c in the low-density limit, where

$$T_c \sim \mu \mathrm{e}^{\pi/(2\sqrt{\mu}a)}$$

as shown in [9]. This result was obtained for a class of integrable interaction potentials with negative scattering length a in the absence of bound states. Thus, we rigorously confirm the *ubiquity of superconducting domes in BCS theory* for a general class of interaction potentials, claimed in [14]. As dicussed in the introduction, this is of particular interest as the presence of a superconducting dome has often been interpreted as an indication of competing orders or some other kind of exotic superconductivity.

3 Proofs

The most important tool for our proofs will be the Birman-Schwinger principle (see [7, 5, 11]). According to this principle, T_c is determined by the fact that for $T = T_c$ the smallest eigenvalue of

$$B_T = V^{1/2} \frac{1}{K_{T,\mu}} |V|^{1/2}$$

equals -1. Here, we used the notation $V(x)^{1/2} = \mathrm{sgn}(V(x))|V(x)|^{1/2}$. The main simplification is that the study of the spectrum of the unbounded operator $K_{T,\mu} + V$ reduces to identifying the singular part of the compact Birman-Schwinger. With this in mind, our proofs will all build on the same convenient decomposition of B_T in a dominant singular term and other error terms. Therefore, let $\mathfrak{F}_\mu^{(d)} : L^1(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1})$ denote the Fourier transform restricted to $\mathbb{S}^{d-1}$ with

$$\left(\mathfrak{F}_\mu^{(d)}\psi\right)(p) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\sqrt{\mu}p\cdot x}\psi(x)\mathrm{d}x\,,$$

which is well-defined by the Riemann-Lebesgue Lemma. Since $V \in L^1(\mathbb{R}^d)$, the multiplication with $|V|^{1/2}$ is a bounded operator from $L^2(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$, and hence $\mathfrak{F}_\mu^{(d)}|V|^{1/2}$ is a bounded operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{S}^{d-1})$. Moreover, one of the key ideas in our proofs is to study the asymptotic behavior of

$$m_\mu^{(d)}(T) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{R}^d} \left(\frac{1}{K_{T,\mu}(p)} - \frac{1}{p^2+\mu}\right) \mathrm{d}p$$

which was done in a similar way for the low-density and weak-coupling limit of the critical temperature and the energy gap (see [8, 9, 11, 16]). One can easily show that

$$m_\mu^{(d)}(T) \sim \mu^{\frac{d-2}{2}} \ln\left(\frac{\mu}{T}\right) \tag{9}$$

as long as $T/\mu \to 0$ (see, e.g., [8]). Using the definitions above, we arrive at our convenient decomposition, which we define as

$$B_T = V^{1/2}\frac{1}{p^2+\mu}|V|^{1/2} + m_\mu^{(d)}(T)\, V^{1/2}\mathfrak{F}_\mu^{(d)\dagger}\mathfrak{F}_\mu^{(d)}|V|^{1/2} + A_{T,\mu}^{(d)}\,, \tag{10}$$

where $A_{T,\mu}^{(d)}$ is such that this holds. For the second term, note that

$$V^{1/2}\mathfrak{F}_\mu^{(d)\dagger}\mathfrak{F}_\mu^{(d)}|V|^{1/2}$$

is isospectral to $\mathcal{V}_\mu^{(d)} = \mathfrak{F}_\mu^{(d)} V \mathfrak{F}_\mu^{(d)\dagger}$. In fact, the spectra agree at first except possibly at 0, but 0 is in both spectra as the operators are compact on an infinite dimensional space.

This second term will be the dominant term, which is how the quantity $e_\mu^{(d)}$ appears in the asymptotic formulae in our theorems. Whereas, the first and third term are negligible error terms in the limit $\mu \to \infty$. Showing this, is the objective of the proofs of Theorem 1, Theorem 2 (b), and Theorem 3. Their main difference lies in the treatment of $A_{T,\mu}^{(d)}$. The proof of Theorem 2 (a) is based on a result of Gontier *et al.* in [6] and will be somewhat different.

A priori, it is not clear, how T_c behaves at high densities. Therefore, before we go to the proofs, let us fix the following

Lemma 1. $T_c = O(\mu)$ *as* $\mu \to \infty$.

Proof. Since $\tanh(t) \leq \min(1, t)$ for $t \geq 0$, we have

$$\begin{aligned} K_{T,\mu} + V &\geq \frac{1}{2}\left(|p^2 - \mu| + 2T\right) + V \\ &\geq \frac{1}{2}\left(p^2 + \mu + 2V\right) + (T - \mu). \end{aligned}$$

The first term is non-negative for sufficiently large μ by the conditions on V. Thus, by Definition 1, we obtain $T_c \leq \mu$. $\square$

In the proofs below, we will in fact show that $T_c = o(\mu)$, so (9) gives the correct asymptotic behavior.

Proof of Theorem 1. As outlined above, the strategy of the proof is to show that the first and the third term in (10) vanish in operator norm in the high-density limit and thus the asymptotic behavior is entirely determined by the spectrum of the operator in the second term. We discuss this in detail now.

For the first term, note that the Fourier transform of $\frac{1}{p^2+\mu}$ is given by $\frac{\mathrm{e}^{-\sqrt{\mu}|x|}}{|x|}$, up to a constant. Thus the Hilbert-Schmidt norm $\|\cdot\|_{\mathrm{HS}}$, which is always an upper bound for the operator norm $\|\cdot\|_{\mathrm{op}}$, is given by

$$\left\| V^{1/2} \frac{1}{p^2 + \mu} |V|^{1/2} \right\|_{\mathrm{HS}}^2 = C \int_{\mathbb{R}^3} \mathrm{d}x \int_{\mathbb{R}^3} \mathrm{d}y \, |V(x)| \frac{\mathrm{e}^{-\sqrt{\mu}|x-y|}}{|x-y|} |V(y)|$$

which vanishes as $\mu \to \infty$ by an application of the dominated convergence theorem in combination with the Hardy-Littlewood-Sobolev inequality (Theorem 4.3 in [17]). Here and the following, we shall use the notation C for generic positive constants, possible having a different value in each appearance.

For the third term, we will heavily use the radiality of V. In fact, since V is radially symmetric, every eigenfunction of $K_{T,\mu}$ and thus B_T will have definite angular momentum and we can focus on $f \in L^2(\mathbb{R}^3)$ of the form $f(x) = f(|x|) Y_\ell^m(\hat{x})$, with a slight abuse of notation, where $\hat{x} = x/|x|$ denotes the unit vector in direction x. Now we aim to bound $\langle f, A_{T,\mu} f \rangle$ uniformly in ℓ (and m). As $A_{T,\mu}$ has integral kernel

$$A_{T,\mu}(x, y) = C V^{1/2}(x) |V(y)|^{1/2} \int_{\mathbb{R}^3} \left(\frac{1}{K_{T,\mu}(p)} - \frac{1}{p^2 + \mu} \right) \left(\mathrm{e}^{\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}\sqrt{\mu}\hat{p}\cdot(x-y)} \right) \mathrm{d}p \,,$$

and using the radial symmetry of V we arrive at

$$\langle f, A_{T,\mu} f \rangle = C \int_0^\infty \mathrm{d}|x| \, |x|^2 \int_0^\infty \mathrm{d}|y| \, |y|^2 \bar{f}(|x|) V^{1/2}(|x|) |V(|y|)|^{1/2} f(|y|) \tag{11}$$

$$\times \int_{\mathbb{R}^3} \mathrm{d}p \left(\frac{1}{K_{T,\mu}(p)} - \frac{1}{p^2 + \mu} \right) \tag{12}$$

$$\times \int_{\mathbb{S}^2} \mathrm{d}\omega(x) \int_{\mathbb{S}^2} \mathrm{d}\omega(y) \overline{Y}_\ell^m(\hat{x}) Y_\ell^m(\hat{y}) \left(\mathrm{e}^{\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}\sqrt{\mu}\hat{p}\cdot(x-y)} \right) . \tag{13}$$

The last line (13) evaluates to

$$16\pi^2 \left(j_\ell(|p||x|) j_\ell(\sqrt{\mu}|y|) - j_\ell(\sqrt{\mu}|x|) j_\ell(|p||y|) \right) \overline{Y}_\ell^m(\hat{p}) Y_\ell^m(\hat{p}) .$$

After performing the angular integration from the second line (12) and writing x and y instead of $|x|$ and $|y|$, respectively, these two lines combine to

$$\int_0^\infty \mathrm{d}p\, p^2 \left(\frac{1}{K_{T,\mu}(p)} - \frac{1}{p^2+\mu} \right) \left(j_\ell(px) j_\ell(py) - j_\ell(\sqrt{\mu}x) j_\ell(\sqrt{\mu}y) \right) . \tag{14}$$

In order to bound this quantity uniformly in ℓ, we need the following properties of the spherical Bessel functions:

(i) uniform boundedness, i.e. $\sup_{\ell\in\mathbb{N}_0} \sup_{x\geq 0} |j_\ell(x)| \leq 1$

(ii) uniform decay, i.e. $\sup_{\ell\in\mathbb{N}_0} \sup_{x\geq 0} |x^{5/6} j_\ell(x)| \leq c$ for some universal constant $c > 0$ (see [13])

(iii) uniform Lipschitz continuity, i.e. $\sup_{\ell\in\mathbb{N}_0} \sup_{x\geq 0} |j_\ell'(x)| \leq 1$. This follows from the uniform boundedness and the recursion relation [1]

$$j_\ell' = \frac{1}{2\ell+1} \left(\ell j_{\ell-1} - (\ell+1) j_{\ell+1} \right) .$$

Using these properties and applying the variable transformation $p \to p/\sqrt{\mu}$ in the integral, we obtain

$$\begin{aligned} |(14)| \leq C\, \mu^{1/3} \int_0^\infty &\left(\frac{|j_\ell(p\sqrt{\mu}x)| + |j_\ell(\sqrt{\mu}x)|}{y^{1/3}} + \frac{|j_\ell(p\sqrt{\mu}y)| + |j_\ell(\sqrt{\mu}y)|}{x^{1/3}} \right) \\ &\times \left| \frac{1}{K_{T/\mu,1}(p)} - \frac{1}{p^2+1} \right| |p-1|^{1/3} (p+p^2)\, \mathrm{d}p \end{aligned}$$

with the aid of the triangle inequality. By employing Hölder for the integrals over x and y in (11), we get

$$|\langle f, A_{T,\mu} f\rangle| \leq C\, \mu^{1/3} \|f\|_2^2 \left\| \frac{V}{|\cdot|^{2/3}} \right\|_1^{1/2} \tag{15}$$

$$\times \int_0^\infty \mathrm{d}p \left| \frac{1}{K_{T/\mu,1}(p)} - \frac{1}{p^2+1} \right| |p-1|^{1/3} (p+p^2)$$

$$\times \int_{\mathbb{R}^3} \mathrm{d}x\, |V(x)| \left(|j_\ell(p\sqrt{\mu}|x|)|^2 + |j_\ell(\sqrt{\mu}|x|)|^2 \right) . \tag{16}$$

Note, that the norm in (15) is finite since $V \in L^{3/2}(\mathbb{R}^3)$. In Lemma 2 below, we show that the last term can be estimated as

$$(16) \leq C \frac{1}{\mu^{1/2}} \left(\frac{1}{p} + 1 \right) .$$

Thus, we arrive at

$$|\langle f, A_{T,\mu} f \rangle| \ \leq C \frac{1}{\mu^{1/6}} \|f\|_2^2 \left\| \frac{V}{|\cdot|^{2/3}} \right\|_1^{1/2} \int_0^\infty \mathrm{d}p \left| \frac{1}{K_{T/\mu,1}(p)} - \frac{1}{p^2+1} \right| |p-1|^{1/3} (1+p)^2 ,$$

where the integral is uniformly bounded as long as $T \leq C\mu$ and we conclude

$$\limsup_{\mu \to \infty} \sup_{0 < T \leq C\mu} \|A_{T,\mu}\|_{\mathrm{op}} = 0 ,$$

since the bound above is uniform in ℓ. Therefore, as long as $T = O(\mu)$, the spectrum of the Birman-Schwinger operator approaches the spectrum of the operator in the second term in (10) as $\mu \to \infty$.

Combining our considerations, we get that, since by assumption $e_\mu^{(3)} < 0$ for $\mu \geq \mu_0$, $T_c > 0$ for all sufficiently large μ. This is because the second term in (10) can be made arbitrarily negative by taking $T \to 0$, whereas the first and the third term are bounded uniformly in $T \leq C\mu$. Thus we get with the aid of Lemma 1 that

$$-1 = \lim_{\mu \to \infty} m_\mu^{(3)}(T_c) e_\mu^{(3)} .$$

In order to obtain (5) by means of (9), it remains to show that $T_c = o(\mu)$. Since it is already shown in Lemma 1 that $T_c = O(\mu)$, we assume that $T_c = \Theta(\mu)$, i.e. there exist $0 < c < C$ such that $c\mu \leq T_c \leq C\mu$. This means that $m_\mu^{(3)}(T_c)$ is of order $\sqrt{\mu}$, which leads to a contradiction since $\sqrt{\mu} e_\mu^{(3)} = o(1)$ by Lemma 2 below. So, (9) implies (5) as desired. □

Lemma 2. *Let $V \in L^{3/2}(\mathbb{R}^3)$. Then*

$$\limsup_{\mu \to \infty} \sqrt{\mu} \sup_{\ell \in \mathbb{N}_0} \int_{\mathbb{R}^3} \mathrm{d}x \, |V(x)| \left(j_\ell(\sqrt{\mu}|x|) \right)^2 = 0 .$$

Proof. We estimate

$$\sqrt{\mu} \sup_{\ell \in \mathbb{N}_0} \int_{\mathbb{R}^3} \mathrm{d}x \, |V(x)| \left(j_\ell(\sqrt{\mu}|x|) \right)^2 \ \leq C \sqrt{\mu} \int_{\mathbb{R}^3} \mathrm{d}x \, |V(x)| \frac{1}{\left(\sqrt{\mu}|x|\right)^{5/3} + 1} , \tag{17}$$

where the inequality follows from property (ii) of the spherical Bessel functions given in the proof of Theorem 1. By using Hölder, we can further bound

$$(17) \leq C \|V - \phi\|_{3/2} \left\| \frac{1}{|\cdot|^{5/3} + 1} \right\|_3 + C \sqrt{\mu} \int_{\mathbb{R}^3} \mathrm{d}x \, |\phi(x)| \frac{1}{\left(\sqrt{\mu}|x|\right)^{5/3} + 1}$$

for any $\phi \in C_0^\infty(\mathbb{R}^3)$. The second term vanishes as $\mu \to \infty$ since $\phi \in C_0^\infty(\mathbb{R}^3)$, the first term can be made arbitrarily small as $C_0^\infty(\mathbb{R}^3)$ is dense in $L^{3/2}(\mathbb{R}^3)$. Thus, we have proven the claim. □

Now, we turn to the proof of upper and lower bounds on the critical temperature for more general interactions in $d = 3$.

Proof of Theorem 2.
Part (a): As a first step, note that $K_{T,\mu}(p) + V(x) \geq 0$ is equivalent to $K_{T/\mu,1}(p) + \frac{1}{\mu}V(x/\sqrt{\mu}) \geq 0$. Then we estimate

$$\begin{aligned} K_{T/\mu,1}(p) &+ \frac{1}{\mu}V(x/\sqrt{\mu}) \\ &\geq \frac{1}{2}\left(|p^2-1| + \frac{2T}{\mu}\right) - \frac{1}{\mu}V_-(x/\sqrt{\mu}) \\ &\geq \frac{1}{2}\left(|p^2-1| + \frac{2T}{\mu} - \frac{2}{\mu}V_-(x/\sqrt{\mu})\left(\mathrm{e}^{-m|x|} + m|x|\right)\right) \\ &\geq \frac{1}{2}\left(|p^2-1| + \frac{2T}{\mu} - \frac{2}{\sqrt{\mu}}\| |V| \cdot | \|_\infty \left(\frac{\mathrm{e}^{-m|x|}}{|x|} + m\right)\right) \end{aligned}$$

for any $m > 0$. By definition of T_c, we have the bound

$$T_c \leq -\frac{\mu}{2}\,\mathrm{infspec}\left(|p^2-1| - \frac{2}{\sqrt{\mu}}\| |V| \cdot | \|_\infty \left(\frac{\mathrm{e}^{-m|x|}}{|x|} + m\right)\right).$$

In [6], Gontier *et al.* obtained an optimized lower bound on infspec$(\cdot)$ by choosing the parameter $m = (\mathrm{const.})\mu^{1/4}\mathrm{e}^{-\sqrt{\frac{\pi}{2\| |V| \cdot | \|_\infty}}\mu^{1/4}}$. Thereby, employing their estimate, which turns into an upper bound by the sign, we arrive at

$$T_c \lesssim \mu \exp\left(-\sqrt{\frac{\pi}{2\| |V| \cdot | \|_\infty}}\mu^{1/4}\right)$$

as given in (6).
Part (b): First, note that the critical temperature, $\tilde{T}_c$, determined by the fact that the smallest eigenvalue of

$$\frac{|o(1)|}{\ln(\mu)}B_T = \lambda B_T$$

equals -1, gives a lower bound on the true critical temperature T_c, as B_T is monotonically increasing in T. Here, $o(1)$ denotes any continuous function that vanishes as $\mu \to \infty$. So, we put ourselves artificially in a "high-density with weak-coupling" situation with weak-coupling parameter $\lambda = \frac{|o(1)|}{\ln(\mu)}$. We will see, that the choice of λ is optimal for the proof of our lower bound.

We employ the same convenient decomposition of the Birman-Schwinger operator in (10). As shown in the proof of Theorem 1, the Hilbert-Schmidt norm of the first term vanishes with the aid of the Hardy-Littlewood-Sobolev inequality (Theorem 4.3 in [17]), indeed even without λ. For the third term, the additional factor of λ will be crucial for estimating its Hilbert-Schmidt norm. In fact, after performing the angular integration, its kernel is given by

$$
\begin{aligned}
A_{T,\mu}(x,y) &= C\, V^{1/2}(x)|V(y)|^{1/2} \\
&\quad \times \int_0^\infty \left(\frac{1}{K_{T,\mu}(p)} - \frac{1}{p^2+\mu} \right) \left(\frac{\sin p|x-y|}{p|x-y|} - \frac{\sin\sqrt{\mu}|x-y|}{\sqrt{\mu}|x-y|} \right) p^2 \,\mathrm{d}p\,.
\end{aligned}
$$

Using that $|\sin a/a - \sin b/b| \le \min(C\frac{|a-b|}{|a+b|}, \frac{1}{a}+\frac{1}{b})$ for $a,b>0$, we can bound

$$
\begin{aligned}
|A_{T,\mu}(x,y)| &\le C|V(x)|^{1/2}|V(y)|^{1/2}\sqrt{1+\frac{1}{|x-y|^2}} \\
&\quad \times \int_0^\infty \left| \frac{1}{K_{T,\mu}(p)} - \frac{1}{p^2+\mu} \right| \left| \frac{p-\sqrt{\mu}}{p+\sqrt{\mu}} \right|^{\varepsilon} \left(\frac{1}{p} + \frac{1}{\sqrt{\mu}} \right)^{1-\varepsilon} p^2 \,\mathrm{d}p \qquad (18)
\end{aligned}
$$

for any $\varepsilon \in (0,1)$. The integral in (18) is bounded by (a constant times) $\frac{\mu^{\varepsilon/2}}{\varepsilon}$ uniformly in $T \le C\mu$. By choosing $\varepsilon = 2/\ln(\mu)$, we get with the aid of the Hardy-Littlewood-Sobolev inequality that

$$
\lambda \sup_{0<T\le C\mu} \|A_{T,\mu}\|_{\mathrm{HS}} \;\le\; C\left(\|V\|_1 + \|V\|_{3/2}\right)|o(1)| \xrightarrow{\mu\to\infty} 0\,.
$$

In order to see $\sqrt{\mu}e_\mu^{(3)} = O(1)$ for general V, we write out the integral kernel of the operator in the second term in (10), which is given by

$$
\left(V^{1/2}\mathfrak{F}_\mu^{(d)\dagger}\mathfrak{F}_\mu^{(d)}|V|^{1/2} \right)(x,y) \;=\; \frac{C}{\sqrt{\mu}} V^{1/2}(x)\frac{\sin\sqrt{\mu}|x-y|}{|x-y|}|V(y)|^{1/2}\,,
$$

and apply the Hardy-Littlewood-Sobolev inequality to conclude that its Hilbert-Schmidt norm is bounded by (a constant times) $\frac{1}{\sqrt{\mu}}$. Thus, involving the additional factor of λ, we get with the same arguments as in the proof of Theorem 1 and by the construction from the beginning of the proof, that

$$
T_c \ge \tilde{T}_c \sim \mu \exp\left(\frac{\ln(\mu)}{|o(1)|\sqrt{\mu}e_\mu^{(3)}(V)} \right)
$$

and conclude (7). $\square$

Remark 1. The integral in (18) always behaves as $\mu^{\varepsilon/2}$ and thus cannot be an error term without the extra factor of λ. This makes the main difference to the proof of Theorem 1, where the additional averaging over the osciallations on the sphere "decouples" x and y and leads to a higher power of μ in the denominator. This will change drastically for $d=2$, where the measure is $p\,\mathrm{d}p$ instead of $p^2\,\mathrm{d}p$.

As the last proof, we turn to the $d=2$ case which is simplified due to the reduced dimension as we will see in the proof.

Proof of Theorem 3. We employ the same decomposition from (10) as used in the proofs of Theorem 1 and 2.

After performing the angular integration, the first term has integral kernel

$$2\pi\, V^{1/2}(x)|V|^{1/2}(y)\int_0^\infty \frac{p}{p^2+\mu} J_0(p|x-y|)\,\mathrm{d}p\,,$$

where J_0 is the 0^{th} order Bessel function of the first kind. Since $|J_0(x)| \le C\frac{1}{x^\beta}$ for any $\beta \in [0,1/2]$, we can bound the integral by

$$\frac{1}{|x-y|^{\frac{2\varepsilon}{1+\varepsilon}}}\int_0^\infty \frac{p^{\frac{1-\varepsilon}{1+\varepsilon}}}{p^2+\mu}\,\mathrm{d}p \;=\; \frac{1}{|x-y|^{\frac{2\varepsilon}{1+\varepsilon}}}\frac{1}{\mu^{\frac{\varepsilon}{1+\varepsilon}}}\int_0^\infty \frac{p^{\frac{1-\varepsilon}{1+\varepsilon}}}{p^2+1}\,\mathrm{d}p \;\le\; C\,\frac{1}{|x-y|^{\frac{2\varepsilon}{1+\varepsilon}}}\frac{1}{\mu^{\frac{\varepsilon}{1+\varepsilon}}}\,.$$

Thus, with the aid of the Hardy-Littlewood-Sobolev (HLS) inequality, we find

$$\left\| V^{1/2}\frac{1}{p^2+\mu}|V|^{1/2}\right\|_{\mathrm{HS}} \le C\frac{1}{\mu^{\frac{\varepsilon}{1+\varepsilon}}}\|V\|_{1+\varepsilon} \to 0 \quad \text{as} \quad \mu\to\infty\,.$$

For the third term, after performing the angular integration, its integral kernel is bounded as

$$\begin{aligned}|A_{T,\mu}(x,y)| \;&\le C\,|V(x)|^{1/2}|V(y)|^{1/2}\\ &\quad\times \int_0^\infty \left|\frac{1}{K_{T,\mu}(p)} - \frac{1}{p^2+\mu}\right| |J_0(p|x-y|) - J_0(\sqrt{\mu}|x-y|)|\,p\,\mathrm{d}p\,.\end{aligned}$$

By using that J_0 is Lipschitz continuous and $|J_0(x)| \le C\frac{1}{x^\beta}$ for any $\beta\in[0,1/2]$, we can further estimate

$$\begin{aligned}|A_{T,\mu}(x,y)| \;&\le C\,\frac{1}{\mu^{\frac{\varepsilon}{1+\varepsilon}}}\,\frac{|V(x)|^{1/2}|V(y)|^{1/2}}{|x-y|^{\frac{2\varepsilon}{1+\varepsilon}}}\\ &\quad\times \int_0^\infty \left|\frac{1}{K_{T/\mu,1}(p)} - \frac{1}{p^2+1}\right| |p-1|^{\frac{1-3\varepsilon}{3(1+\varepsilon)}} \left(\frac{1}{\sqrt{p}}+1\right)^{\frac{1+3\varepsilon}{3(1+\varepsilon)}} p\,\mathrm{d}p\,,\end{aligned}$$

where the integral is bounded uniformly in $T \le C\mu$. Using again the HLS inequality, we conclude

$$\limsup_{\mu\to\infty}\ \sup_{0<T\le C\mu}\|A_{T,\mu}\|_{\mathrm{HS}} = 0\,.$$

Therefore, by the same arguments as in the proof of Theorem 1 and using Lemma 1, we get

$$-1 = \lim_{\mu\to\infty} m_\mu^{(2)}(T_c)e_\mu^{(2)}\,.$$

Since $e_\mu^{(2)} = o(1)$ as $\mu\to\infty$ by the Riemann-Lebesgue Lemma (see the Definition of $\mathcal{V}_\mu^{(d)}$ in (2)), we have, similarly to the proof of Theorem 1, that $T_c = o(\mu)$ by involving Lemma 1. So, (9) implies (8) as desired. □

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