Largest Eigenvalue of Rank one Perturbation of Gaussian ensemble

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Chapter 1

Introduction

The purpose of this work is to present a proof of a special case of Theorem 2.1 in [1] (see Theorem 1 below). Our proof will follow closely the proof of Theorem 2.1 in [1] but we try to present it in a more accessible way. Theorem 2.1 in [1] characterizes the high dimensional limit of the extremal eigenvalues of some random matrix $X$ which is perturbed by some deterministic matrix $A$. It states that in the high dimensional limit the matrix $M = X + A$ has eigenvalues outside the spectrum of $X$ iff $A$ is "big enough". Further it gives the explicit values of these eigenvalues outside the spectrum of $X$.

For simplicity we will focus on the Gaussian case namely $X$ being a Gaussian unitary ensemble (GUE) in the complex case and a Gaussian orthogonal ensemble (GOE) in the real case instead of a general Wigner matrix. Further we will also restrict ourselves to the case of $A$ being a rank one projection instead of a finite rank projection. Our convention for the definition of a GUE (resp. GOE) is the following. A GUE (resp. GOE) is a collection of Hermitian matrices $W = W(N) \in \mathbb{C}^{N \times N}$ (resp. $W \in \mathbb{R}^{N \times N}$) such that $W_{ii}$ for $1 \leq i \leq N$ and $\sqrt{2} \Re(W_{ij}), \sqrt{2} \Im(W_{ij})$ for $1 \leq i < j \leq N$ (resp. $\frac{1}{\sqrt{2}} W_{ii}$ for $1 \leq i \leq N$ and $W_{ij}$ for $1 \leq i < j \leq N$) have centered Gaussian distribution with variance $\sigma^2$ and are all independent. We write the rank one projection as $A = \theta a a^*$ with $a = a(N) \in \mathbb{C}^N$, $\|a\| = 1$ (resp. $a \in \mathbb{R}^N$) and $\theta \in \mathbb{R}$. For ease of notation we will also write $X = \frac{1}{\sqrt{N}} W$.

Our goal is to prove the following.

Theorem 1 (Theorem 2.1 in [1]). Let $M = M(N) = X + A$ and denote by $\lambda_k$ the $k$-th largest eigenvalue. Then for any joint realization of $(M(N))_{N \in \mathbb{N}}$ the following statements hold almost surely

$$\lim_{N \to \infty} \lambda_1(M) = \begin{cases} 2\sigma & \text{if } \theta \leq \sigma \\ \theta + \frac{\sigma^2}{\sigma} & \text{if } \theta > \sigma \end{cases} \quad (1.1)$$

and for any fixed $k > 1$

$$\lim_{N \to \infty} \lambda_k(M) = 2\sigma. \quad (1.2)$$

Similarly

$$\lim_{N \to \infty} \lambda_N(M) = \begin{cases} 2\sigma & \text{if } \theta \geq -\sigma \\ \theta + \frac{\sigma^2}{\sigma} & \text{if } \theta < -\sigma \end{cases} \quad (1.3)$$

and for any fixed $k > 1$

$$\lim_{N \to \infty} \lambda_{N-k}(M) = -2\sigma. \quad (1.4)$$

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We want to briefly explain some intuition about the result. The semicircle law states that in the Wigner case \((A = 0)\) the empirical distribution of the eigenvalues of \(M\) converges almost surely to the semicircle distribution \(\mu_{sc}\) given by

\[
\mu_{sc} \sim \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} 1_{[-2\sigma,2\sigma]}(x).
\]

A pedagogical reference is [2]. It can be further shown (Theorem 2.12 in [6]) that in the Wigner case for any fixed \(k \in \mathbb{N}\) we have that almost surely the extremal eigenvalues \(\lambda_1(M), \ldots, \lambda_k(M)\) and \(\lambda_{N-k}(M), \ldots, \lambda_N(M)\) converge to \(2\sigma\) and \(-2\sigma\) respectively as \(N \to \infty\). Thus the result of Theorem 1 can be understood in the following way. If \(|\theta| \leq \sigma\) then in the high dimensional limit the extremal eigenvalues of \(M\) are the same as in the case \(A = 0\). If \(|\theta| > \sigma\) then we get exactly one eigenvalue outside the support of the semicircle distribution. It is also useful to note that \(|\theta + \frac{\sigma^2}{2\theta}| > 2\sigma\) iff \(|\theta| > \sigma\).

Our main tool in proving Theorem 1 will be the Stieltjes transform. For \(z \in \mathbb{C} \setminus \mathbb{R}\) we denote the resolvent of \(M\) by \(G(z) = (zI - M)^{-1}\) and define

\[
g(z) = \text{tr} G(z),
\]

where \(\text{tr} = \frac{1}{N} \text{Tr}\) is the normalized trace.

We also denote by

\[
g_\sigma(z) = \mathbb{E}[(z - s)^{-1}] = \int_{\mathbb{R}} \frac{1}{z - x} d\mu_{sc}(x)
\]

the Stieltjes transform of a random variable \(s\) with semicircle distribution.

We will use \(C\) to denote a generic constant the value of which may change from line to line. Similarly we let \(P\) denote a generic polynomial that may change from line to line. We will also say that a quantity \(\Delta(N, z)\) is \(O(z/N)\) for some \(p \in \mathbb{N}\) if for some \(C, l\)

\[
|\Delta(z)| \leq C \frac{1}{N^p} (1 + |z|)^l (1 + |\text{Im}(z)|)^{-l},
\]

To make notation simpler we will be writing the RHS as \(\frac{1}{N^p} (1 + |z|)^l P(|\text{Im}(z)|)^{-l}\). In our convention always sure convergence refers to a probability space where all \((M_N)_{N \in \mathbb{N}}\) are realized jointly. Since we will be deriving the almost sure convergence through the Borel-Cantelli Lemma, the convergence results holds for any joint realization, so we do not specify an explicit realization.

This work is structured as follows. In chapter 2 we state some properties of the resolvent \(G\) and the Stieltjes transform of the semicircle law \(g_\sigma\). We also introduce two important tools, namely multivariate versions of the Poincaré inequality and Stein’s Lemma. Our goal in chapter 3 is to obtain the "Master equation"

\[
g(z) = g_\sigma(z) + \frac{1}{N} L_\sigma(z) + O(z/N^2)
\]

with an explicit \(L_\sigma\) given in (3.15). This gives us an explicit expression for the deviation of \(g\) from \(g_\sigma\). In chapter 4 we can then use this to show that almost surely

\[
\lim_{N \to \infty} \text{Spect}(M) \subseteq K_\sigma = [-2\sigma, 2\sigma] \cup \{\theta + \frac{\sigma^2}{\theta}\}.
\]

In chapter 5 we deduce Theorem 1 from (1.6) by first considering the case of small \(\sigma\) and then obtaining the general case by a rescaling argument.
Chapter 2

A few tools

We have the following useful properties of $G(z)$ that are immediate to check.

**Lemma 2.** Let $G = (zI - M)^{-1}$ be the resolvent of a Hermitian (resp. symmetric) matrix $M$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Then

1. \[ \|G(z)\| \leq |\Im(z)|^{-1}, \] where $\|\cdot\|$ denotes the operator norm. In particular it holds that $|G_{ij}(z)| \leq |\Im(z)|^{-1}$. \hspace{1cm} (2.1)

2. We have\[ \frac{1}{N} \sum_{i,j=1}^{N} |(GG)_{ij}|^2 = \frac{1}{N} \text{Tr}(G^*G^*GG) \leq |\Im(z)|^{-4}. \] \hspace{1cm} (2.2)

3. The derivative of $G$ with respect to $M$ is given by \[ G'B = GBG \] for any matrix $B$.

4. For any $z \in \mathbb{C}$ such that $|z| > \|M\|$ \[ \|G\| \leq \frac{1}{|z| - \|M\|}. \] \hspace{1cm} (2.4)

We will also need the following properties of $g_\sigma(z)$. All of the statements apart from (2.6) can be easily deduced from (2.6) and the definition of the Stieltjes transform. One way to obtain (2.6) is to explicitly compute $g_\sigma(z)$ for $z \in \mathbb{C} \setminus \mathbb{R}$ using the explicit definition in (1.5).

**Lemma 3.** The following holds

1. \[ g_\sigma \text{ is analytic on } \mathbb{C} \setminus [-2\sigma, 2\sigma]. \] \hspace{1cm} (2.5)

2. For all $z$ such that $\Im(z) \neq 0$

   \(a\) \quad \sigma^2 g_\sigma^2(z) - z g_\sigma(z) + 1 = 0, \hspace{1cm} (2.6)

   \(b\) \quad |g_\sigma(z)| \leq |\Im(z)|^{-1}, \hspace{1cm} (2.7)

   \(c\) \quad |g_\sigma(z)^{-1}| \leq |z| + \sigma^2 |\Im(z)|^{-1}, \hspace{1cm} (2.8)

   \(d\) \quad |g_\sigma'(z)| = \left| \int \frac{1}{(z-x)^2} d\mu_{sc}(x) \right| \leq |\Im z|^{-2}, \hspace{1cm} (2.9)
(e) \[ \Im(g_\sigma(z))\Im(z) < 0, \] \hspace{1cm} (2.10)

(f) \[ \left| \frac{1}{ag_\sigma - z + \theta} \right| \leq |\Im z|^{-1} \text{ for all } a > 0, \theta \in \mathbb{R}. \] \hspace{1cm} (2.11)

3. For all \( z \) such that \( |z| > 2\sigma \)

(a) \[ |g_\sigma(z)| \leq \frac{1}{|z| - 2\sigma}, \] \hspace{1cm} (2.12)

(b) \[ |g'_\sigma(z)| = \left| \int \frac{1}{(z - x)^2} d\mu_\text{sc}(x) \right| \leq \frac{1}{(|z| - 2\sigma)^2}, \] \hspace{1cm} (2.13)

(c) \[ |g_\sigma(z)^{-1}| \leq |z| + \frac{\sigma^2}{|z| - 2\sigma}. \] \hspace{1cm} (2.14)

The Gaussian measure satisfies the following Poincaré inequality. Let \( \mu \) denote the Gaussian measure of the entries of \( W \). Then for any \( f \in H^1(\mathbb{R}, \mu) \cap C^1(\mathbb{R}) \) we have

\[ \text{Var}(f) \leq C\sigma^2 \int |f|^2 \, d\mu, \] \hspace{1cm} (2.15)

where \( \text{Var}(f) = \text{Var}(f(X)) \) and \( X \sim \mathcal{N}(0, \sigma^2) \). Since we do not need the dependence on \( \sigma \) in this inequality we will suppress it in the constant. The Poincaré inequality generalizes to

**Lemma 4.** We have for any complex valued function \( f \) on \( \mathbb{R}^{N^2} \) (resp. \( \mathbb{R}^{N(N+1)/2} \)) such that both \( f \) and \( \nabla f \) are polynomially bounded

\[ \text{Var}(f(M)) \leq \frac{C}{N} \mathbb{E}[\|\nabla f(M)\|^2_2], \]

where \( \|M\|_2 \) denotes the Frobenius norm of a matrix.

We refer to Theorem 3.20 in [4] for a proof of Lemma 4 (and (2.15)) in the case \( A = 0 \). It is straightforward to deduce Lemma 4 from Theorem 3.20 in [4] by applying Theorem 3.20 for \( \tilde{f} \) defined by \( f(M) = \tilde{f}(X) \).

We also have the following generalization of Stein’s Lemma

**Lemma 5.** Let \( \Phi \) be a \( C^1 \) function on the space of Hermitian (resp. symmetric) matrices. Then for any deterministic Hermitian (resp. symmetric) matrix \( H \)

\[ \mathbb{E}[\Phi'(X) \cdot H] = \frac{N}{\sigma^2} \mathbb{E}[\Phi(X) \text{Tr}(XH)] \]

as long as both sides are well defined.

Note that Lemma 5 is just a reformulation of the multivariate version of Stein’s Lemma. The multivariate version is obtained by coordinate wise integration by parts as in the one dimensional version of Stein’s Lemma.
Chapter 3

Master equation

The goal of this section is to establish Lemma 9 which states that

\[ g(z) = g_\sigma(z) + \frac{1}{N} L_\sigma(z) + O_z\left(\frac{1}{N^2}\right) \]  

for an explicit \( L_\sigma \) (defined in (3.15)). This will then allow us to obtain information on \( \text{Spect}(M) \) in the next chapter. We now briefly explain the strategy. The main idea is to first prove that \( g(z) \) satisfies

\[ \sigma^2 g(z) - zg(z) + 1 + \Delta(z) = O_z\left(\frac{1}{N^2}\right) \]  

for some appropriate quantity \( \Delta(z) \) that is \( O_z(N^{-1}) \). Then we use that \( g_\sigma \) satisfies an equation similar to (3.2) (namely (2.6)) to obtain an asymptotic estimate of the form

\[ g(z) = g_\sigma(z) + \tilde{\Delta}(z) + O_z\left(\frac{1}{N^p}\right) \quad \text{for } p \in \{1, 2\}. \]  

We will use a two step bootstrap argument. In the first step we will prove a crude version of (3.2) in Lemma 6 that will allow us to obtain (3.3) for some \( \tilde{\Delta}(z) \) and \( p = 1 \) in Lemma 7. We can then obtain a stronger version of (3.2) in Lemma 8, which then allows us to prove (3.1) in Lemma 9.

Lemma 6. It holds that

\[ \left| \sigma^2 g^2(z) - zg(z) + 1 + \frac{1}{N} \mathbb{E}[\text{Tr}(G(z)A)] \right| \leq \frac{P(|\Im(z)|^{-1})}{N^2}. \]  

Proof. Let \( \{E_{ij}\}_{1 \leq i,j \leq N} \) be the canonical basis of the space of \( N \times N \) matrices. Applying Lemma 5 to \( \Phi(X) = G_{ij} = (zI - X - A)^{-1} \) and \( H = E_{ij} \) for any \( 1 \leq i,j \leq N \) we obtain

\[ \mathbb{E}[G_{ii}G_{jj}] = \frac{N}{\sigma^2} \mathbb{E}[G_{ij}X_{ij}]. \]

Taking the normalized sum \( \frac{1}{N^2} \sum_{i,j} \) gives

\[ \mathbb{E}[\text{tr}(G)^2] \leq \frac{1}{\sigma^2} \mathbb{E}[\text{tr}(GX)]. \]  

Writing

\[ GX = (zI - X - A)^{-1}(X + A - zI - A + zI) = -I - GA + zG, \]

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\[ (3.5) \text{ implies } \mathbb{E}[\text{tr}(G)^2] + \frac{1}{\sigma^2}(1 + \mathbb{E}[\text{tr}(GA)] - z \mathbb{E}[\text{tr}(G)]) = 0. \]

To conclude (3.4) it now suffices to prove that

\[ \text{Var}(\text{tr}(G)) \leq C |\Re(z)|^{-4}. \tag{3.6} \]

Applying Lemma 4 with \( f(\Phi(M)) = \text{tr}(G) \) and using the identity (2.3) for \( B = E_{ij} \) we get

\[ \text{Var}(\text{tr}(G)) \leq \frac{C}{N^2} \mathbb{E} \left[ \sum_{i,j} |(GG)_{ij}|^2 \right] \leq \frac{C}{N^2} |\Re(z)|^{-4} \]

so (3.6) holds.

**Lemma 7.** For any \( z \in \mathbb{C} \) with \( \Re(z) > 0 \)

\[ |g(z) - g_\sigma(z)| \leq (|z| + C) \frac{P(|\Re(z)|^{-1})}{N}. \tag{3.7} \]

**Proof.** We first note that since \( A \) is rank one we have

\[ |\mathbb{E}[\text{tr}(G(z)A)]| \leq \|G(z)\| \leq |\Re(z)|^{-1}, \]

so it follows from Lemma 6 that

\[ |\sigma^2g^2 - zg + 1| \leq \frac{P(|\Re(z)|^{-1})}{N}. \tag{3.8} \]

We also define

\[ O = \left\{ z \in \mathbb{C} |\Re(z) > 0, \frac{P(|\Re(z)|^{-1})}{N} (|z| + \sigma^2 |\Re(z)|^{-1}) |\Re(z)|^{-1} < \frac{1}{4} \right\}, \]

where \( P_O \) chosen such that (3.4) and (3.8) hold for \( P = P_O \). It is easily seen that \( O \) is nonempty for large enough \( N \). Now let \( z \in O \). Noticing that

\[ \frac{P_O(|\Re(z)|^{-1})}{N} < \frac{1}{4}, \]

it follows from (3.8) that

\[ |g(z)| |\sigma^2g(z) - z| \geq \frac{1}{2} \]

which, using (2.1), implies

\[ \frac{1}{|g(z)|} \leq 2(|z| + \sigma^2 |\Re(z)|^{-1}). \tag{3.9} \]

Thus we can define

\[ \Lambda(z) = \sigma^2g(z) + \frac{1}{g(z)}. \]
Then from (3.8) and (3.9) it follows that
\[ |\Lambda(z) - z| \leq \frac{P(|\Im(z)|^{-1})}{N^2} |z| + \sigma^2 |\Im(z)|^{-1}. \tag{3.10} \]
Combining (3.10) with the fact that by construction of \( O \)
\[ \frac{P(|\Im(z)|^{-1})}{N^2} |z| + \sigma^2 |\Im(z)|^{-1} \leq \frac{|\Im(z)|}{2} \]
we obtain
\[ |\Im(\Lambda(z)) - \Im(z)| \leq |\Lambda(z) - z| \leq \frac{|\Im(z)|}{2}, \]
so
\[ \Im(\Lambda(z)) > \frac{|\Im(z)|}{2} > 0. \tag{3.11} \]
Lastly we want to show that
\[ g(z) = g_\sigma(\Lambda(z)). \tag{3.12} \]
Let \( O' = \{ z \in O | |\Im(z)| > 2\sigma \} \), then for any \( z \in O' \) we have from (3.11) and (2.8) that \( g_\sigma(\Lambda(z)) \neq 0 \). Therefore we obtain from (2.6)
\[ \sigma^2 g_\sigma(\Lambda(z)) + \frac{1}{g_\sigma(\Lambda(z))} = \Lambda(z) = \sigma^2 g(z) + \frac{1}{g(z)}. \]
Rearranging and multiplying both sides by \( g_\sigma(\Lambda(z))g(z) \) we obtain
\[ \sigma^2 g_\sigma(\Lambda(z))g(z) (g_\sigma(\Lambda(z)) - g(z)) = g_\sigma(\Lambda(z)) - g(z). \]
We now show that \( \sigma^2 |g_\sigma(\Lambda(z))g(z)| < 1 \). We have \( |g(z)| \leq |\Im(z)|^{-1} \leq \frac{1}{\sigma_\sigma} \) and due to (2.7), (3.11) also \( |g_\sigma(\Lambda(z))| \leq |\Im(\Lambda(z))|^{-1} < \frac{1}{2} \) so (3.12) holds on \( O' \). Since \( O' \) is open and \( O \) is connected by analytic continuation (3.12) holds on all of \( O \).
Now (3.12)
\[ |g(z) - g_\sigma(z)| = |E[(z - s)^{-1}(\Lambda(z) - s)^{-1}](\Lambda(z) - z)| \]
\[ \leq |\Im(z)|^{-1} |\Im(\Lambda(z))|^{-1} |\Lambda(z) - z|, \]
where \( s \) is distributed according to the semicircle law with variance \( \sigma^2 \). Combining (3.10), (3.11) now shows that (3.7) holds true on \( O \).
It remains to consider the case \( z \notin O \). For such a \( z \) we have by definition of \( O \) that
\[ |\Im(z)|^{-1} \leq \frac{4P(|\Im(z)|^{-1})}{N^2} |z| + \sigma^2 |\Im(z)|^{-1} |\Im(z)|^{-2}. \tag{3.13} \]
Since \( |g(z)|, |g_\sigma(z)| \leq |\Im(z)|^{-1} \) due to (2.1) and (2.7) respectively we observe that (3.7) holds also for \( z \notin O \).

\[ \square \]

**Lemma 8.** For any \( z \in \mathbb{C} \) with \( \Im(z) > 0 \)
\[ \left| \sigma^2 g(z) - z g(z) + \frac{1}{N} E_\sigma(z) \right| \leq (|z| + C) \frac{P(|\Im(z)|^{-1})}{N^2}, \]
where \( E_\sigma(z) := \frac{\theta}{z - \sigma \tau g_\sigma(z) - \sigma} \) and \( \theta \) is the nonzero eigenvalue of \( A \).
Proof. We argue similarly as in Lemma 6. Applying Lemma 5 with $\Phi = G_{il}$ and $H = E_{jl}$ we obtain
\[
E[G_{ij}G_{il}] = \frac{N}{\sigma^2} E[G_{il}X_{ij}],
\]
so by taking the normalized sum over $l$ we get
\[
\frac{1}{N} \sum_l E[G_{ij}G_{il}] = \frac{1}{\sigma^2} E[(GX)_{ij}],
\]
which can be rearranged as
\[
\sigma^2 E[G_{ij} \text{tr}(G)] = E[(GX)_{ij}].
\]
Writing
\[
GX = (zI - X - A)^{-1}(X + A - zI - A + zI) = -I - GA + zG,
\]
we have for any $i, j$
\[
h_{ij} := \sigma^2 E[G_{ij} \text{tr}(G)] + \delta_{ij} - z E[G_{ij}] + E[(GA)_{ij}] = 0.
\]
Now recalling that $A = \theta a a^*$ with $||a|| = 1$ we define $\alpha = \sum_{i,j} \bar{a}_i a_j G_{ij}$. Noting that
\[
\sum_{i,j} \bar{a}_i a_j (GA)_{ij} = (a, GAa) = \theta \alpha,
\]
we have
\[
0 = \sum_{i,j} \bar{a}_i a_j h_{ij} = \sigma^2 E[\alpha \text{tr}(G)] + 1 + (\theta - z) E[\alpha]. \tag{3.14}
\]
By Jensen’s inequality we have
\[
|E[\alpha(\text{tr}(G) - g)]| \leq E[|\alpha(\text{tr}(G) - g)|^2]^{1/2} = O_z\left(\frac{1}{N}\right),
\]
since $\alpha$ is bounded and $\text{Var}(\text{tr}(G)) = O_z(N^{-1})$ due to (3.6). We can combine this with (3.14) to obtain
\[
E[\alpha](\sigma^2 g(z) + \theta - z) + 1 = O_z\left(\frac{1}{N}\right).
\]
Now it follows from Lemma 7 that $E[\alpha](\sigma^2 g_\sigma(z) + \theta - z) + 1 = O_z(N^{-1})$ so due to (2.11)
\[
\text{Tr}(GA) = \theta E[\alpha] = \frac{\theta}{z - \sigma^2 g_\sigma(z) - \theta} + O_z\left(\frac{1}{N}\right).
\]
Combining this with Lemma 7 finishes the proof. \hfill \square

Lemma 9. For any $z \in \mathbb{C} \setminus \mathbb{R}$
\[
\left| g_\sigma(z) - g(z) + \frac{1}{N} L_\sigma(z) \right| = O_z\left(\frac{1}{N^2}\right),
\]
where
\[
L_\sigma(z) = g_\sigma(z)^{-1} g_\sigma(z) E_\sigma(z) = g_\sigma(z)^{-1} \mathbb{E}[(z - s)^{-2}] E_\sigma(z) \tag{3.15}
\]
and $s$ is distributed according to the semicircle law with variance $\sigma^2$. 8
Proof. We may assume that \( \text{Im}(z) > 0 \), since writing out the definitions of \( g_\sigma(z), g(z), L_\sigma(z) \) we observe that 
\[ g_\sigma(\bar{z}) = \bar{g}(z), g(\bar{z}) = \bar{g}(z), L_\sigma(\bar{z}) = \bar{L}_\sigma(z). \]
If \( z \in \mathcal{O} \)
\[
\left| g_\sigma(z) - g(z) + \frac{1}{N} L_\sigma(z) \right| = \left| g_\sigma(z) - g_\sigma(\Lambda(z)) + \frac{1}{N} L_\sigma(z) \right|
\leq \left| \mathbb{E}[(z-s)^{-1}(\Lambda(z) - s)^{-1}(\Lambda(z) - z) + g_\sigma(z)(z-s)^{-2} E_\sigma(z)] \right|
+ \mathbb{E} \left[ |(z-s)^{-1}((z-s)^{-1} - (\Lambda(z) - s)^{-1})| \right] \left| \frac{1}{N} g_\sigma(z) E_\sigma(z) \right|
\leq 2 |\text{Im}(z)|^{-2} \left| \Lambda(z) - z + \frac{1}{N} g_\sigma^{-1}(z) E_\sigma(z) \right|
+ \frac{P(|\text{Im}(z)|^{-1})}{|z|} \left| \Lambda(z) - z \right| |z| + C. \tag{3.16}
\]
Here we have used (2.7), (3.11), (2.8) and
\[
|E_\sigma(z)| \leq P(|\text{Im}(z)|^{-1}), \tag{3.18}
\]
which follows from (2.11). By (3.10) it follows that the term in (3.11) is \( O_\varepsilon(N^{-2}) \). For the term in (3.16) we have
\[
\left| \Lambda(z) - z + \frac{1}{N} g_\sigma^{-1}(z) E_\sigma(z) \right| = O_\varepsilon(N^{-2}).
\]
It remains to consider the case \( z \notin \mathcal{O} \). By definition of \( \mathcal{O} \) we have that \( z \notin \mathcal{O} \) implies
\[
1 < (|z| + C) \frac{P(|\text{Im}(z)|^{-1})}{N},
\]
so it is enough to show that
\[
\left| g_\sigma(z) - g(z) + \frac{1}{N} L_\sigma(z) \right| = O_\varepsilon(1).
\]
But this holds since Lemma 7 implies that \( |g_\sigma(z) - g(z)| = O_\varepsilon(N^{-1}) \) and (2.8) and (3.18) imply that \( L_\sigma(z) = O_\varepsilon(1) \).
Chapter 4

The spectrum of $M$

Using the bounds from the last chapter we are now able to prove the following

**Theorem 10.** Let $\rho := \theta + \frac{\sigma^2}{\theta}$ where $\theta$ is the nonzero eigenvalue of $A$ and

$$K_\sigma := [-2\sigma, 2\sigma] \cup \{\rho\}.$$  

Then almost surely

$$\lim_{N \to \infty} \text{Spect}(M) \subseteq K_\sigma.$$  

This theorem will be the main ingredient for the proof of Theorem 1 in the next chapter. To prove Theorem 10 we will first show that $L_\sigma(z)$ is the Stieltjes transform of a distribution $\mu_\sigma$, the support of which is contained in $K_\sigma$. Then Lemma 9 will give us a bound on the expected number of eigenvalues outside of $K_\sigma$. Finally a bound on the variance of this number will allow us to deduce Theorem 10.

For the first step will we use the following characterization which we do not prove.

**Theorem 11** ([5]). Let $\mu$ be a distribution (in the sense of generalized functions) on $\mathbb{R}$ with compact support $K$ and denote by $l(z) = \mu(\frac{1}{z-x})$ its Stieltjes transform. Then $l$ is analytic on $\mathbb{C}\setminus\mathbb{R}$ and can be extended to $\mathbb{C}\setminus K$. Moreover $l$ satisfies

i) $\lim_{|z| \to \infty} l(z) = 0$.

ii) There exists a compact set $K$ and $n \in \mathbb{N}$, such that for any $z \in \mathbb{C}\setminus\mathbb{R}$

$$|l(z)| \leq C \max\{1, \text{dist}(z, K)^{-n}\}.$$  

iii) For any test function $\varphi \in C^\infty_c(\mathbb{R}, \mathbb{R})$

$$\mu(\varphi) = -\frac{1}{\pi} \lim_{y \to 0^+} \Im \int_\mathbb{R} \varphi(x) l(x + iy) \, dx.$$  

**Conversely if $K \subset \mathbb{R}$ is compact and $l$ is an analytic function on $\mathbb{C}\setminus K$ satisfying i), ii) , then $l$ is the Stieltjes transform of a distribution $\mu$ on $\mathbb{R}$ with supp($\mu$) $\subset K$. Moreover supp $\mu$ is precisely the set of singular points of $l$.**

This allows us to prove

**Lemma 12.** We have that $L_\sigma(z)$ defined in (3.15) is the Stieltjes transform of a distribution $\mu_\sigma$, the support of which is contained in $K_\sigma$. 

Proof. We will first show that \( L_\sigma \) satisfies \( i) \) and \( ii) \) in Theorem 11. Recall that

\[
L_\sigma (z) = g_\sigma (z)^{-1} g'_\sigma (z) \frac{\theta}{z - \sigma^2 g_\sigma - \theta}.
\]

It follows from (2.12), (2.13), (2.14) that \( L_\sigma \) satisfies \( i) \). From (2.8), (2.9), (2.11) it follows that \( ii) \) holds locally with \( n = 4 \). Since \( L_\sigma \) vanishes at \( \infty \) it follows that \( ii) \) holds (globally).

Due to Theorem 10 we now only need to show that \( L_\sigma \) has no singular points in \( K^c_\sigma \). By (2.6) we can rewrite

\[
L_\sigma (z) = g_\sigma (z)^{-1} g'_\sigma (z) \frac{\theta}{1/g_\sigma (z) - \theta}.
\]

Using that for \( x \in \mathbb{R} \setminus [-2\sigma, 2\sigma] \)

\[
g_\sigma (x) = \frac{x}{2\sigma^2} (1 - \sqrt{1 - 4\sigma^2 / x^2}),
\]

we see that

\[
1/g_\sigma (x) - \theta = 0 \quad \iff \quad x = \rho_\sigma.
\]

Thus the statement of the Lemma is immediate from the explicit form of \( L_\sigma (x) \).

We will now show that

**Lemma 13.** For any test function \( \varphi \in C^\infty_c (\mathbb{R}, \mathbb{R}) \) we have

\[
\mathbb{E}[\text{tr}(\varphi(M))] = \int \varphi \, d\mu_{sc} - \frac{1}{N} \mu_{\sigma}(\varphi) + O(\frac{1}{N^2}). \tag{4.1}
\]

Thus for any real valued smooth \( \varphi \) such that \( \varphi \) is constant outside of a compact set and \( \text{supp} \varphi \cap K_\sigma = \emptyset \) there is a constant \( C_\varphi \) such that almost surely

\[
|\text{tr}(\varphi(M))| \leq C_\varphi N^{-\frac{3}{4}} \quad \text{for almost every } N. \tag{4.2}
\]

**Proof.** Let \( r(z) = g(z) - g_\sigma (z) + \frac{1}{N} L_\sigma (z) \). Since \( g(z), -g_\sigma (z), \frac{1}{N} L_\sigma (z) \) are all the Stieltjes transform of some distribution \((g(z), -g_\sigma (z))\) by definition and \( \frac{1}{N} L_\sigma (z) \) due to Lemma 12), \( r \) is the Stieltjes transform of the sum of these distributions. Observing that the definitions of \( g, g_\sigma, L_\sigma \) imply \( r(\tau) = r(z) \) we can write the inverse Stieltjes transform of \( r \) as

\[
\mathbb{E}[\text{tr}(\varphi(M))] - \int \varphi \, \mu_{sc} + \frac{1}{N} \mu_{\sigma}(\varphi) = -\frac{1}{\pi} \lim_{y \to 0^+} \text{Im} \int_{\mathbb{R}} \varphi(x) r(x + iy) \, dx.
\]

We know from Lemma 9 that

\[
|r(z)| \leq \frac{1}{N^2} (|z| + C)^\alpha P(|\text{Im}(z)|^{-1}) \tag{4.3}
\]

for some \( \alpha \in \mathbb{R}_+ \). It can be shown that this implies

\[
\limsup_{y \to 0^+} \left| \int_{\mathbb{R}} \varphi(x) r(x + iy) \, dx \right| \leq C \frac{1}{N^2}.
\]

We only sketch the proof here. Define for \( p \in \mathbb{N}_+ \)

\[
I_p(z) = \frac{1}{(p-1)!} \int_0^\infty r(z + t) t^{p-1} \exp(-t) \, dt.
\]
It is easy to check that
\[ I_1(z) - I_1'(z) = r(z) \]
and for \( p \geq 2 \)
\[ I_p(z) - I_p'(z) = I_{p-1}. \]
Let \( D \) denote the operator that takes the derivative of a function. Using these identities we can iteratively use integration by parts to obtain
\[ \int_{\mathbb{R}} \varphi(x) r(x + iy) \, dx = \int_{\mathbb{R}} ((1 + D)^p \varphi) I_p(x + iy) \, dx. \] (4.4)

Then one can show that (4.3) implies
\[ \lim_{r \to \infty} \int_{[r,r+ir]} r(z + \tilde{z}) \tilde{z}^{p-1} \exp(-\tilde{z}) \, d\tilde{z} = 0 \]
so by Cauchy’s integral Theorem along the contour \([0,r] \cup [r+ir,0] \cup [r+ir,0]\), where \([z_1,z_2]\) denotes the line segment going from \(z_1\) to \(z_2\)
\[ I_p(z) = \lim_{r \to \infty} \int_{[0,r+ir]} r(z + \tilde{z}) \tilde{z}^{p-1} \exp(-\tilde{z}) \, d\tilde{z}. \]

Recall that by our convention the bound in Lemma 9 is to be understood as
\[ |r(z)| \leq (|z| + C)^\frac{p([\text{Im}(z)]^{-1})}{N^2}. \]

Now we can choose \( p = k + 1 \) where \( k \) is the degree of \( P \). It is then straightforward to check that
\( I_p(z) \) is bounded on any compact set, so (4.1) follows from (4.4).

To prove (4.2) below we will prove a bound on the variance of \( \text{tr}(\varphi(M)) \) namely
\[ \text{Var}(\text{tr}(\varphi(M))) = O(N^{-4}) \] (4.5)
for any \( \varphi \) such that \( \varphi \) is constant outside of a compact set and \( \text{supp} \varphi \cap K_\sigma = \emptyset \). Assume that we have already proven this bound and let \( Z_N = \text{tr}(\varphi(M)) \), \( \Omega_N = \{Z_N > N^{-4/3}\} \). From (4.1) and (4.5) we obtain
\[ E[|Z_N|^2] = O(N^{-4}). \]

Now
\[ P(\Omega_N) \leq \int_{\Omega} |N^{4/3} Z_N(\omega)|^2 \, dP(\omega) \leq N^{8/3} E[|Z_N|^2] = O(N^{-4/3}), \]
so (4.2) follows from the Borel-Cantelli Lemma.

It remains to prove (4.5). We will deduce this from Lemma 4. As in Lemma 4 we will identify matrices with vectors in \( \mathbb{R}^{N^2} \) (resp. \( \mathbb{R}^{N(N+1)/2} \)) and denote by \( \|\cdot\|_2 \) the Frobenius norm. Write \( \varphi = c + \psi \) with \( c \in \mathbb{R} \) and \( \psi \in C_c^\infty \). For \( g(M) := \text{tr}(\varphi(M)) \) we obtain from Lemma 4
\[ \text{Var}(g(M)) \leq \frac{C}{N} E[\|\nabla g(M)\|_2^2]. \]

We have for any Hermitian (resp. symmetric) matrix \( B \)
\[ \text{Tr}(\nabla\psi(M) \cdot B) = \text{Tr}(\psi'(M) \cdot B). \]
This can be proven by showing this identity for polynomials and then extending it to all bounded functions by approximating them with polynomials. We have

\[
\|\nabla g(M)\|_2^2 = \sup_{\|B\|_2=1} |\langle \nabla g(M), B \rangle|^2
\]

\[
= \sup_{\|B\|_2=1} \left| \frac{d}{dt} \bigg|_{t=0} g(M + tB) \right|^2.
\]

Since the trace is linear we get

\[
\left| \frac{d}{dt} \bigg|_{t=0} g(M + tB) \right|^2 = |\text{tr}(\nabla \psi(M) \cdot B)|^2
\]

\[
= |\text{tr}(\psi'(M) \cdot B)|^2
\]

\[
= \frac{1}{N^2} |\text{tr}(\psi'(M) \cdot B)|^2
\]

\[
\leq \frac{1}{N^2} \text{Tr}(\psi'(M) \cdot \psi'(M)) \text{Tr}(B \cdot B)
\]

\[
= \frac{1}{N} \text{tr}((\psi')^2(M)).
\]

Where we have used the Cauchy-Schwartz inequality for the trace and the fact that \(\text{Tr}(B \cdot B) = \|B\|_2^2 = 1\). Putting everything together we obtain

\[
\text{Var}(g(M)) \leq \frac{C}{N} \|\nabla g(M)\|_2^2 \leq \frac{C}{N^2} \text{tr}((\psi')^2(M)),
\]

so now (4.5) follows from (4.1) since \(\psi\) vanishes on \(K_\sigma\).

Now we can quickly deduce Theorem 10.

Proof of Theorem 10. For \(\epsilon > 0\) let \(\varphi\) be a smooth function that vanishes on \(K_\sigma\) and is equal to 1 on the complement of \(K_\sigma + (-\epsilon, \epsilon)\). Lemma 13 implies that almost surely \(\lim_{N \to \infty} \text{Tr}(\varphi(M)) = 0\). Since the number of eigenvalues outside of \(K_\sigma + (-\epsilon, \epsilon)\) is an integer and bounded by \(\text{Tr}(\varphi(M))\) that number must be equal to 0 almost surely as \(N \to \infty\). Since \(\epsilon\) was arbitrary we are done.
Chapter 5

Proof of Theorem [1]

We are now able to deduce Theorem [1] from Theorem [10]. The main tool that we will use is the min-max-principle:

**Lemma 14.** Let $H$ be a self-adjoint (resp. symmetric) matrix on $V = \mathbb{C}^N$ (resp. $V = \mathbb{R}^N$). Denote by $\lambda_k$ the $k$-th largest eigenvalue of $H$. Then

$$\lambda_k = \min_{\dim(U)=k-1} \max_{\|v\|=1} \langle v, Hv \rangle. \quad (5.1)$$

Consequently for any self-adjoint (resp. symmetric) matrices $H, \tilde{H}$ we have

$$\lambda_{j+k-1}(H + \tilde{H}) \leq \lambda_j(H) + \lambda_k(\tilde{H}) \quad (5.2)$$

and for $j + k \geq N + 1$

$$\lambda_j(H) + \lambda_k(\tilde{H}) \leq \lambda_{j+k-N}(H + \tilde{H}). \quad (5.3)$$

We refer to Section 12.1 in [3] for a proof of (5.1). Statements (5.2) and (5.3) are easily deduced from (5.1).

The structure of the proof of Theorem [1] is the following. For $\sigma \geq \theta$ Theorem [1] follows directly from Lemma [14], the semicircle law and Theorem [10]. For $\sigma < \theta$ again combining Lemma [14], the semicircle law and Theorem [10] will imply that Theorem [1] is true for small $\sigma$. We will then argue by contradiction that if Theorem [1] holds for some $\sigma$ it is also holds for some slightly bigger $\sigma$, which then allows us to conclude. Here the semicircle law refers to the statement that in the Wigner case ($A = 0$) the empirical distribution of the eigenvalues of $M$ converges almost surely to the semicircle distribution $\mu_{sc}$. We are now ready for the

**Proof of Theorem [1]** As an immediate consequence of Lemma [14] we obtain (1.2) and (1.4) from the semicircle law, by setting $H = A$ and $\tilde{H} = W$ and using that $A$ has rank one. Thus we only need to prove (1.1) and (1.3). By replacing $A$ with $-A$ and using that $W$ has the same distribution as $-W$ we may w.l.o.g assume that $\theta > 0$ (recall that $A = \theta aa^*$. In this case (1.3) is also an immediate consequence from Lemma [14] and the semicircle law. Thus we only need to prove (1.1).

We first show that we may assume that $\theta > \sigma$. If $\theta \leq \sigma$ then from Theorem [10] it follows that almost surely

$$\limsup_{N \to \infty} \lambda_1(M) \leq 2\sigma.$$ 

But since due to (1.2) we always have

$$2\sigma = \lim_{N \to \infty} \lambda_2(M) \leq \lim_{N \to \infty} \lambda_1(M),$$

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it follows that (1.1) is true for $\theta \leq \sigma$.

Thus we can assume that $\theta > \sigma$. Due to Theorem 10 we only need to show that almost surely

$$\liminf_{N \to \infty} \lambda_1(M) > 2\sigma,$$

as if we are given such a lower bound then Theorem 10 implies that almost surely

$$\lim_{N \to \infty} \lambda_1(M) = \rho_\sigma$$

i.e. (1.1) holds.

Let now $\theta$ be fixed and define $\Sigma$ be the set of all $\sigma < \theta$ for which (5.4) fails ($M, W, A$ still depend on this $\sigma$). We need to show that $\Sigma = \emptyset$. Assume that $\Sigma \neq \emptyset$, then it makes sense to define

$$\sigma_0 := \inf \Sigma.$$

Since Lemma 14 implies that (5.4) holds if $\sigma$ is small enough we have $\sigma_0 > 0$. Now let $\sigma = \sigma_0 + \frac{\epsilon}{2}$ for some (small) $\epsilon > 0$ let $M_\epsilon = M - \epsilon W$. From the definition of $\sigma_0$ it follows that almost surely

$$\lim_{N \to \infty} \lambda_1(M_\epsilon) = \rho_{\sigma - \epsilon}.$$

Since $M = M_\epsilon + \epsilon W$ it follows from the min-max principle and the semicircle law that almost surely

$$\liminf_{N \to \infty} \lambda_1(M) \geq \rho_{\sigma - \epsilon} - 2\sigma \epsilon = \theta + \frac{\sigma^2}{\theta} + \epsilon \left( \frac{-2 + \epsilon}{\theta} - 2\sigma \right).$$

Letting $\epsilon \to 0$ we obtain

$$\liminf_{N \to \infty} \lambda_1(M) \geq \theta + \frac{\sigma^2}{\theta} > 2\sigma,$$

where we have used that by assumption $\theta > \sigma$. Thus $\sigma = \sigma_0 + \frac{\epsilon}{2}$ satisfies (5.4). But this contradicts the definition of $\sigma_0$ so we must have $\Sigma = \emptyset$ which completes the proof. 

$\square$
Bibliography


