EXISTENCE AND UNIQUENESS OF MINIMIZERS FOR THE NONLINEAR HARTREE EQUATION

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1. Introduction

In [2], Elliot H. Lieb studies the problem of minimizing the functional

$$\mathcal{E}(\phi) = \int |\nabla\phi|^2 dx - \int\int \frac{|\phi(x)|^2|\phi(y)|^2}{|x-y|} dxdy, \tag{1.1}$$

over functions in $H^1(\mathbb{R}^3)$ with fixed L^2 norm. The problem is completely solved in the paper, showing existence and uniqueness of a minimizer (up to phase and translations). Moreover, it is also shown that if ϕ is a minimizer then ϕ must be symmetric decreasing with respect to some point $x \in \mathbb{R}^3$.
Here we study the same problem working with the space $H_0^1(B_R)$. We will get existence and uniqueness of the minimizer (this time only up to phase).

2. Notations and Known Facts

The following are basic notations and definitions that we will use throughout this survey.

- $T(\phi) = \int |\nabla\phi|^2 dx$.
- $W(\phi) = \int\int |\phi(x)|^2|\phi(y)|^2|x-y|^{-1}dxdy$.
- $E(\lambda) = \inf\{\ \mathcal{E}(\phi) \mid \phi \in H^1(\mathbb{R}^3),\ \ ||\phi||_2 \leq \lambda\}$.
- $E(\lambda, R) = \inf\{\ \mathcal{E}(\phi) \mid \phi \in H_0^1(B_R),\ \ ||\phi||_2 \leq \lambda\}$.

We will also make use of the machinery of symmetric decreasing rearrangements of functions. If f is a measurable non-negative function on $\mathbb{R}^3$ we will denote its symmetric decreasing rearrangement by f^*. If f is measurable from $\mathbb{R}^3$ to $\mathbb{C}$ we define the symmetric decreasing rearrangement of f to be $f^* = |f|^*$.
We will need the following properties of symmetric decreasing rearrangements.

Proposition 2.1. *Let f be a measurable function and let f^* be its symmetric decreasing rearrangement. Then:*

- $\forall\ 1 \leq p \leq \infty$ *we have* $||f||_p = ||f^*||_p$.
- *For any three measurable functions f, g and h we have*

$$\left|\int\int f(x)g(x-y)h(y)dxdy\right| \leq \int\int f^*(x)g^*(x-y)h^*(y)dxdy. \tag{2.1}$$

- *If additionally g is a symmetric strictly decreasing non-negative function, then the previous inequality is strict (provided the right-hand side is finite), unless f and g are equicentered symmetric decreasing functions.*
- *If $\phi \in H^1(\mathbb{R}^3)$ then $\phi^* \in H^1(\mathbb{R}^3)$ and $T(\phi) \geq T(\phi^*)$.*

(2.1) *is called Riesz Inequality.*

3. EXISTENCE AND OTHER PRELIMINAR RESULTS

Following the approach of [2], it is easy to show that $\mathcal{E}$ is well defined on $H^1_0(B_R)$ for every $R > 0$ and that $E(\lambda, R)$ is always finite. Indeed functions in $H^1_0(B_R)$ can be considered as a subset of $H^1(\mathbb{R}^3)$, by extending them to be zero outside of B_R, and thus the estimates for $\mathcal{E}$ developed in [2] still hold in our case.
It is pretty easy to show $E(\lambda)$ is always negative (the Gaussian, renormalized to have L^2 norm equal to λ, always works in $\mathbb{R}^3$). On the other hand, we claim that if $\lambda^2 R$ is sufficiently small, then $E(\lambda, R) \geq 0$.
Indeed, first of all we can use the Sobolev embedding theorem in $\mathbb{R}^3$ to have that, for any $\phi \in H^1_0(B_R)$,

$$T(\phi) = T(\tilde{\phi}) \geq K||\tilde{\phi}||_6^2 = K||\phi||_6^2. \tag{3.1}$$

Where we have denoted by $\tilde{\phi}$ the extension of ϕ to the whole $\mathbb{R}^3$.
Next we estimate W. The function $|x|^{-1}$ can be written as

$$|x|^{-1} = h_1^r(x) + h_2^r(x),$$

where $h_1^r(x) = |x|^{-1}\chi_{B_r}(x)$ and thus $h_1^r \in L^{3/2}$ and $h_2^r \in L^\infty$. This implies that

$$W(\phi) = \int_{B_R}\int_{B_R} |\phi(x)|^2|\phi(y)|^2(h_1^r(x-y) + h_2^r(x-y)) \leq$$
$$\leq ||h_1^r||_{3/2}||\phi||_2^2||\phi||_6^2 + ||h_2^r||_\infty||\phi||_2^4\,\chi_{[0,2R]}(r).$$

This implies that, working under the assumption that $||\phi||_2 = \lambda$, we have

$$\mathcal{E}(\phi) \geq K||\phi||_6^2 - ||h_1^r||_{3/2}||\phi||_2^2||\phi||_6^2 - ||h_2^r||_\infty||\phi||_2^4\,\chi_{[0,2R]}(r) \geq$$
$$||\phi||_6^2(K - Cr\lambda^2) - (1/r)\lambda^4\,\chi_{[0,2R]}(r).$$

Usually, on $\mathbb{R}^3$, to show boundedness of $\mathcal{E}$, one takes r small enough so that the first term is positive and can be thrown away (otherwise one has no direct control for it), paying this choice with an increase in the absolute value of the second term. On B_R though, one has to pay on the second term only if $r \leq 2R$, this implies that, if $\lambda^2 R$ is small enough, then r can be bigger then $2R$ and one gets away without paying (and with the claim).

This fact is actually pretty relevant since it changes the "structure" of the problem. To be more precise, it's easy to see that as long as $\mathcal{E}(\phi)$ is negative, then increasing the norm of ϕ lowers $\mathcal{E}$. This, together with the fact that $E(\lambda)$ is negative for any λ on $\mathbb{R}^3$, allows us to study the problem over functions with L^2-norm smaller or equal to λ since then the minimizers will always have norm as big as possible (namely λ). On the other hand, if $\lambda^2 R$ is too small then $E(\lambda, R)$ is going to be zero and $\phi \equiv 0$ is going to be a minimizer.

We highlight that, so far, is only clear that, once λ is fixed, for R sufficiently small then $\mathcal{E}$ is always positive. This tells us nothing on the behaviour for big R, in particular we are not even assured that $\mathcal{E}$ is *ever* negative. Next Lemma assures us that for every λ there exists R such that $E(\lambda, R) < 0$.

Lemma 3.1. *For every $\lambda > 0$, $\exists\ R(\lambda)$ such that $E(\lambda, R(\lambda))$ is strictly negative (of course this will imply that $E(\lambda, R) < 0$ for every $R \geq R(\lambda)$ since, by definition, $E(\lambda, R)$ is decreasing in R, being the infimum over nested sets).*

Proof. We fix $\lambda > 0$ for the whole proof. We are trying to minimize $\mathcal{E}$ over functions in $H^1_0(B_R)$ with fixed L^2-norm λ. The following is a simple but key observation: we claim that studying the problem on $H^1_0(B_R)$ is the same as to try to minimize $R^{-2}\mathcal{E}_R$

over functions in $H^1_0(B_1)$ with L^2-norm equal to λ. Here $\mathcal{E}_R(\cdot) = T(\cdot) - RW(\cdot)$. This follows easily by taking any function ϕ in $H^1_0(B_1)$ and considering $\bar{\phi}(x) = R^{-3/2}\phi(R^{-1}x)$. This transformation is a bijection between $H^1_0(B_1)$ and $H^1_0(B_R)$ and leaves the L^2-norm unchanged, moreover, we have:

$$\mathcal{E}(\bar{\phi}) = R^{-2}\mathcal{E}_R(\phi).$$

This proves the claim. Now we observe that $\mathcal{E}_R(\phi) \to -\infty$, as $R \to \infty$, for any ϕ in $H^1_0(B_1)$ s.t. $W(\phi) \neq 0$ and thus, for R big enough, the infimum over functions in $H^1_0(B_1)$ with L^2-norm equal to λ of $\mathcal{E}_R$ (we define this quantity as $E_R(\lambda, 1)$) must be negative. This completes the proof since the sign of $E(\lambda, R)$ is the same of the sign of $E_R(\lambda, 1)$. □

To summarize we have shown that for each λ we have that $E(\lambda, R)$ is zero for a while and then starts decreasing. This definition of $E(\lambda, R)$, though, is not satisfactory for small R, since the problem becomes trivial. Thus, from now on, we will define $E(\lambda, R)$ as before only for $R \geq R(\lambda)$, whereas, when the usual $E(\lambda, R) = 0$, we will define it to be the infimum of $\mathcal{E}$ over functions in $H^1_0(B_R)$ with L^2-norm *exactly* equal to λ.

Having understood this behaviour, we will now prove the following usefull Lemma.

Lemma 3.2. *Fix λ and R. Then $T(\phi)$ is uniformly bounded if $\mathcal{E}(\phi)$ is close to $E(\lambda, R)$.*

Proof. We know, by the previous calculations, splitting $T(\phi)$ in 2 parts and omitting $\chi_{[0,2R]}(r)$, that:

$$\mathcal{E}(\phi) \geq T(\phi)/2 + ||\phi||_6^2(K/2 - Cr\lambda^2) - (1/r)\lambda^4.$$

This implies, after the appropriate choice $r = \frac{K}{2C\lambda^2}$:

$$\mathcal{E}(\phi) \geq T(\phi)/2 - (2C/K)\lambda^6.$$

This basically complete the proof, recalling that $\mathcal{E}(\phi)$ is close to $E(\lambda, R)$ and thus bounded. □

Now we use the nice properties of symmetric decreasing rearrangements. We observe that, by 2.1, we immediately get that, if f is any function in $H^1_0(B_R)$, then $\mathcal{E}(f) \geq \mathcal{E}(f^*)$ and, moreover, the symmetric decreasing rearrangement leaves the L^2-norm unchanged. This implies that rearranging always gets us closer to finding a minimizer, or at least does no harm (this will imply that we will be able to take the minimizing sequence realizing $E(\lambda, R)$ to be composed of symmetric decreasing functions - even when $\lambda^2 R$ is small). To get this we have not used the strict Riesz Inequality, which implies the following stronger result.

Lemma 3.3. *If $\phi \in H^1_0(B_R)$, $||\phi||_2 = \lambda$ and $\mathcal{E}(\phi) = E(\lambda, R)$ (i.e. ϕ is a minimizer), then $|\phi|$ is symmetric decreasing with respect to some $x \in B_R$.*

All these things are pretty straightforward to show (the reader can find more details in [2], we highlight that everything works exactly the same on $H^1_0(B_R)$ and $H^1(\mathbb{R}^3)$). We also highlight that the last result can (and will) be improved, indeed we expect the symmetry point x to be 0 and the minimizing function to be strictly positive in $H^1_0(B_R)$ (if $x \neq 0$ then the function has to be zero on a portion of B_R).

Also the existence of a minimizer can actually be obtained exactly in the same way in $H^1_0(B_R)$ and in $H^1(\mathbb{R}^3)$.

Proposition 3.4 (Existence of minimizers)**.** *For every λ, R there exists a function ϕ such that $\phi \in H^1_0(B_R)$, $||\phi||_2 = \lambda$, ϕ is symmetric decresing with respect to 0 and such that $\mathcal{E}(\phi) = E(\lambda, R)$.*

Proof. We will only sketch the proof for completeness, for more details we refer to [2] since the arguments used are exactly the same. As stated above, we have the freedom of choosing a minimizing sequence to be symmetric decreasing with respect to 0. By Lemma 3.2 we also know that the $H^1_0(B_R)$-norm of the sequence is uniformly bounded and thus the sequence is weakly convergent, up to subsequences, to some $\phi \in H^1_0(B_R)$. This also implies L^2-weak convergence and thus, by lower semicontinuity, we get $\liminf T(\phi_j) \geq T(\phi)$ and $||\phi||_2 \leq \lambda$ (we highlight that the first inequality is telling us that the limit function ϕ behaves well with respect to T, whereas the second is telling us that it behaves well with respect to the L^2-norm - at least when $\lambda^2 R$ is not too small - and thus we only need to show that ϕ also behaves well with respect to W to get that it is a minimizer).
We highlight that in the following, by abuse of notation, we will write $h(r)$ instead of $h(|x|)$ for any spherical function h. To continue the proof, we use the fact that the sequence is composed of spherically symmetric and decreasing functions, which are also uniformly bounded in L^2 and L^6 norms (the latter is due to Lemma 3.2 and Sobolev embedding theorem), to retrieve that $\phi_j(r) \leq f(r)$, for some spherical decreasing function f with good properties (order $|x|^{-1/2}$ near zero). Through this bound we recover pointwise convergence (here, again, the fact that the functions are all spherically decreasing is of key importance) to some function that can easily be shown to be again ϕ. This finally implies the convergence of $W(\phi_j)$ to $W(\phi)$ (by dominated convergence with the domitation given by f), showing that ϕ must be a minimizer. Finally we highlight that $||\phi||_2 = \lambda$ (again for dominated convergence) and thus we have also shown the existence for a minimizer in the case when $\lambda^2 R$ is small. □

We conclude this section, which basically just consists of known results adapted to the case of $H^1_0(B_R)$. We highlight that, so far, the situation in $H^1_0(B_R)$ and $H^1(\mathbb{R}^3)$ appears to be basically the same. In the next section we will show uniqueness, together with some other small results, in what is the most novel and interesting part of this work: indeed, the first relevant differences between the two cases wil arise.

4. Uniqueness

We begin with some preliminary results/observations.

Proposition 4.1 (Variational Characterization)**.** *If $\phi \in H^1_0(B_R)$, $||\phi||_2 = \lambda$ and $\mathcal{E}(\phi) = E(\lambda, R)$ then ϕ satisfies the following (distributional) equation on B_R (i.e. integrated against any function in $C^\infty_0(B_R)$):*

$$(-\Delta + V_\phi)\phi = e\phi. \tag{4.1}$$

Here $V_\phi(x) = -2\int |\phi(y)|^2|x-y|^{-1}dy$ and e is some Lagrange multiplier. Moreover, if $\phi \in H^1_0(B_R)$, not necessarily minimizing, satisfies (4.1) *in the distributional sense on B_R, then ϕ is $C^\infty(B_R)$ and thus a solution of* (4.1) *in the strong sense.*

Proof. The first part is standard. We just take $g \in C^\infty_0(B_R)$ and consider $\mathcal{E}(\phi + \varepsilon g)$ and compute the derivative at $\varepsilon = 0$. We observe that $e < 0$ if $\mathcal{E}(\phi) < 0$. Indeed, multiplying by ϕ and integrating we get:

$$e = \lambda^{-2}(\mathcal{E}(\phi) - W(\phi)).$$

Thus we immediately have that if $\mathcal{E}(\phi) < 0$ also $e < 0$. For the second part we will just sketch the bootstrap argument used to derive regularity. First we observe that V_ϕ is in L^p for any $4 \leq p \leq \infty$ and, since it is a convolution between ϕ and $|x|^{-1}$, V_ϕ will also share any smoothness that ϕ has. This implies that ϕV_ϕ will be as regular as ϕ (the summability will come from the fact that V_ϕ is in L^∞ and the smoothness from the fact that ϕ and V_ϕ share the same smoothness). Thus, if we suppose any regularity for ϕ we then have that $\Delta\phi$ is just as regular. If we now make use of elliptic regularity theory, the bootstrap argument is complete. □

We can also get some interesting and very important (as most of the following results we rely heavily on some kind of scaling argument) "scaling" properties of the problem.

Proposition 4.2. *Minimizing $\mathcal{E}$ on $H_0^1(B_R)$ under the constraint $||\cdot||_2 = \lambda$ is the same as:*

- *minimizing $R^{-2}\mathcal{E}_R$ on $H_0^1(B_1)$ under the constraint $||\cdot||_2 = \lambda$,*
- *minimizing $R^{-3}\mathcal{E}$ on $H_0^1(B_1)$ under the constraint $||\cdot||_2 = \sqrt{R}\lambda$.*

Proof. Actually we already proved the first part in the proof of Lem 3.1. For the second part the idea is the same, we take any function $\phi \in H_0^1(B_R)$ and we transform it into $\bar{\phi}(x) := R^2\phi(Rx) \in H_0^1(B_1)$. This transformation is a bijection and transforms the L^2-norm by multypling it with a factor $\sqrt{R}$, moreover $\mathcal{E}(\bar{\phi}) = R^3\mathcal{E}(\phi)$ and this completes the proof. □

The following is a very important result, since the main difference between the $H_0^1(B_R)$ and $H^1(\mathbb{R}^3)$ cases will be implied, i.e. uniqueness only up to phase and not up to translations.

Proposition 4.3. *If $\phi \in H_0^1(B_R)$, $||\phi||_2 = \lambda$ and $\mathcal{E}(\phi) = E(\lambda, R)$ (i.e. ϕ is a minimizer), then $|\phi|$ is symmetric decreasing with respect to 0 and is strictly positive on B_R.*

Proof. We already know by Lemma 3.3 that ϕ must be symmetric with respect to some $y \in B_R$. We suppose by contradiction that $y \neq 0$ then, since the problem is invariant for translation, $\psi(x) := \phi(x - y)$ is symmetric decreasing with respect to 0 and is still a minimizer, moreover $supp(\psi) \subset B_{R-|y|}$. We want to show that this is a contradiction. We have to distinguish between the two cases $e \geq 0$ and $e < 0$.
If $e \geq 0$ we consider a function $g \in C_0^\infty(B_R)$ such that $g \equiv 1$ on $B_{R-|y|}$. Then, by (4.1), we have

$$\langle(-\Delta + V_\psi)\psi, g\rangle = \langle e\psi, g\rangle$$
$$\Rightarrow \cancel{\langle\psi, \Delta g\rangle} + \langle V_\psi\psi, g\rangle = \langle e\psi, g\rangle.$$

Here the first term vanishes because g is constant on $supp(\psi)$. This is a contradiction since $\psi \geq 0$ and $g \equiv 1$ on $supp(\psi)$ whereas $V_\psi \leq 0$. This implies that the term on the left is strictly negative, whereas the term on the right is nonnegative (using $e \geq 0$).
If $e < 0$ we observe that since $supp(\psi) \subset\subset B_R$ then (4.1) is satisfied not only against any function in $C_0^\infty(B_R)$ but also for any function in $C_0^\infty(\mathbb{R}^3)$, i.e. ψ satisfies (4.1) in $\mathbb{R}^3$. Moreover, since $e < 0$, then we can invert $(-\Delta - e)$, getting

$$\psi = Y_e * (\psi V_\psi),$$

Where $Y_e(x) = (4\pi|x|)^{-1}exp(-|e|^{1/2}|x|)$. This implies that $\psi > 0$ on $\mathbb{R}^3$ which is of course a contradiction since $supp(\psi) \subset\subset B_R$ □

Before actually showing uniqueness, we still need some preparation. We define

$$I(\phi) = \int |\phi|^2/|x| dx.$$

For any function $\phi \in H_0^1(B_R)$, $I(\phi)$ is finite, since $\phi \in L^2 \cap L^6$. Moreover we can always scale ϕ with the rescaling $\bar{\phi}(x) = (\lambda_1/\lambda)^4 \phi((\lambda_1/\lambda)^2 x)$ (similar to the rescaling in 4.2). $\bar{\phi}$ will have L^2-norm equal to λ_1, will be supported on $B_{R(\lambda/\lambda_1)^2}$ and will be such that $I(\bar{\phi}) = (\lambda_1/\lambda)^4 I(\phi)$. We highlight that, if $\lambda_1 > \lambda$, then we are concentranting the function and making it more peaked, whereas if $\lambda_1 < \lambda$ then we are flattening the function and making it more distributed. Moreover, if ϕ satisfies (4.1) with some $e = e(\phi)$, then it can be shown with some calculations that $\bar{\phi}$ will satisfy (4.1) on $B_{R(\lambda/\lambda_1)^2}$ with $e(\bar{\phi}) = (\lambda_1/\lambda)^4 e(\phi)$.
Now we write V_ϕ in a more clever way (using the approach in [2] which applies Newton's theorem):

$$V_\phi(x) = \int_0^{|x|} K(|x|, s)|\phi(s)|^2 ds - 2I(\phi) := U_\phi(x) - 2I(\phi).$$

Here $K(|x|, s) = 8\pi s^2(s^{-1} - |x|^{-1}) \geq 0$. Thus (4.1) can be written as:

$$(-\Delta + U_\phi)\phi = [e(\phi) + 2I(\phi)]\phi. \tag{4.2}$$

We define $\nu(\phi) := [e(\phi) + 2I(\phi)]$. Of course we can do the same for $\bar{\phi}$, getting the same equation with $\nu(\bar{\phi}) = (\lambda_1/\lambda)^4 \nu(\phi)$. We also highlight that by multypling by ϕ and integrating we get that $\nu(\phi)$ is always positive, since $U_\phi(x) \geq 0$.
Finally we observe that if ϕ is radial decreasing with respect to the origin then so is $\bar{\phi}$ and moreover (4.2) can be written in the form:

$$\left[-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + U_\phi(r)\right]\phi(r) = \nu(\phi)\phi(r), \quad r \geq 0. \tag{4.3}$$

We are now ready to show our main result.

Theorem 4.4 (Uniqueness up to phase)**.** *For any λ, R there exists a unique positive function ϕ minimizing $\mathcal{E}(\cdot)$ over functions in $H_0^1(B_R)$ with fixed L^2-norm equal to λ.*

Proof. As we have already shown, any positive minimizer must be symmetric decreasing with respect to the origin. The proof will be split in two parts. In the first part we will show that, fixed λ and R, any minimizer has to solve the equation in the form (4.3) with the same eigenvalue. In the second part we will then show uniqueness of solutions for that equation (once the eigenvalue is fixed).
(1) We suppose by contradiction that there exist ϕ_1 and ϕ_2, both minimizers of the problem with parameters λ and R, such that $\nu(\phi_1) \neq \nu(\phi_2)$. Without loss of generality we assume $\nu(\phi_1) > \nu(\phi_2)$. We then rescale ϕ_2 to increase its eigenvalue and make it equal to $\nu(\phi_1)$. By previous observations, we then need to consider a rescaling $\bar{\phi_2}$ of ϕ_2 with increased L^2-norm, supported on a smaller ball (of radius $R^* < R$) and more peaked. Now, both $\bar{\phi_2}$ and ϕ_1 satisfy the equation:

$$-v''(r) - \frac{2}{r}v'(r) + U_v(r)v(r) = \nu(\phi_1)v(r), \quad r \in [0, R^*]. \tag{4.4}$$

We want to show that this is a contradiction and we will do this mainly following the approach in [1]. By standard fixed point arguments, we deduce that equation (4.4) has a unique local C^2 solution for given initial data $v(0)$, $v'(0)$ and the solution exists up to some maximal radius $\bar{R} \geq R^*$. We first observe that since both functions are radially decreasing (and smooth) then we will have that $\phi_1'(0) = \bar{\phi_2}'(0) = 0$. If now $\phi_1(0) = \bar{\phi_2}(0)$ then we get a contradiction since $\bar{\phi_2}(r) \to 0$ as $r \to R^*$, whereas $\phi_1(r) > 0$ for all $r < R$, by Proposition 4.3. Now we claim that the two functions can not intersect even in $[0, R^*)$. Suppose $\bar{\phi_2}(0) > \phi_1(0)$ then by elementary calculations and using the equation we get the following "Wronskian-type" identity:

$$r^2[\phi_1(r)\bar{\phi_2}'(r) - \phi_1'(r)\bar{\phi_2}(r)] = \int_0^r s^2\phi_1(s)\bar{\phi_2}(s)(U_{\bar{\phi_2}}(s) - U_{\phi_1}(s))ds. \tag{4.5}$$

By continuity $\bar{\phi_2}(r) > \phi_1(r)$ at least for small r. If we suppose that the two functions first intersect in $\hat{r}$ then we have a contradiction evaluating (4.5) at $\hat{r}$ since the left-hand side must be negative, whereas the right-hand side must be strictly positive at $\hat{r}$ (since $U_{\bar{\phi_2}}(r) - U_{\phi_1}(r) > 0$ for any $r \in (0, \hat{r})$). If instead $\bar{\phi_2}(0) < \phi_1(0)$ we will get a similar contradiction with reversed signs. Thus the claim is proved. Now we have that either $\phi_1 > \bar{\phi_2}$ or $\phi_1 < \bar{\phi_2}$ in $(0, R^*)$, but in both cases we get a contradiction. Indeed, the first case is not possible since $||\phi_1||_{L^2(B_{R^*})} < ||\phi_1||_{L^2(B_R)} < ||\bar{\phi_2}||_{L^2(B_{R^*})}$, whereas the second case is not possible since, as said before, $\bar{\phi_2}(r) \to 0$ as $r \to R^*$, whereas $\phi_1(r) > 0$ for all $r < R$. This proves (1).

(2) Now (1) is proved and thus ϕ_1 and ϕ_2 both satisfy the equation

$$-v''(r) - \frac{2}{r}v'(r) + U_v(r)v(r) = \nu v(r), \quad r \in [0, R). \tag{4.6}$$

We can suppose, without loss of generality that $\nu = 1$ (if not we just rescale the functions). By using the same arguments as in part (1) we easily get that one of the two functions is strictly bigger than the other in $[0, R)$, which is a contradiction since they have the same L^2-norm. The proof of the theorem is complete. □

5. Convergence Results

So far we have shown that, for any λ and any R, there exists a unique positive minimizer of $\mathcal{E}$ over functions in $H_0^1(B_R)$ with L^2-norm equal to λ. We fix λ and we consider ϕ_R to be the sequence of said minimizers. Each one of these minimizer satisfies the equation:

$$(-\Delta + U_{\phi_R})\phi_R = \nu(R)\phi_R,$$

on $H_0^1(B_R)$. On the other hand, as shown in [2], there exists a unique positive Φ, symmetric decreasing with respect to the origin, which minimizes $\mathcal{E}$ over functions in $H^1(\mathbb{R}^3)$ with L^2-norm equal to λ. This Φ satisfies the equation:

$$(-\Delta + U_\Phi)\Phi = \mathcal{V}\phi.$$

In this section we will show that $\nu(R) \to \mathcal{V}$, $\phi_R \to \Phi$ (both in $L^2(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$ norms, as well as pointwise) and finally we will also get that $E(\lambda, R) \to E(\lambda)$.
We begin with a preliminary result.

Lemma 5.1. *The sequence $\nu(R)$ is strictly decreasing and $\nu(R) \geq \mathcal{V}$ for any R.*

Proof. Let $R_1 < R_2 < \infty$ and let ϕ_1, ϕ_2 be the minimizers on $H_0^1(B_{R_1})$ and $H_0^1(B_{R_2})$. First of all we observe that $\nu_1 := \nu(R_1) \neq \nu_2 := \nu(R_2)$ because this would lead to a contradiction using the same arguments used in the previous section, the fact that $||\phi_1|| = ||\phi_2|| = \lambda$ and the fact that $supp(\phi_1) = B_{R_1} \subset supp(\phi_2) = B_{R_2}$. If, by contradiction with the thesis, we have $\nu_2 > \nu_1$ we then consider $\bar{\phi_2}$, obtained from ϕ_2 by rescaling and such that $\nu(\bar{\phi_2}) = \nu_1$. Since, as highlighted before, ν is always positive and since we know how ν

scales when rescaling functions, we also have that $\bar{\phi_2}$ is obtained by flattening ϕ_2 and making its support bigger. This latter fact implies a contradiction because, using the usual arguments, we know that either $\bar{\phi_2} = \phi_1$ or one of the two functions is strictly bigger than the other and none of these options make sense since $supp(\phi_1) = B_{R_1} \subset supp(\phi_2) \subset supp(\bar{\phi_2})$ and $||\bar{\phi_2}||_2 < \lambda = ||\phi_1||_2$. This implies that we necessarily have $\nu_1 > \nu_2$ as required. It is worth mentioning that in this case, with a similar argument as above, we do not reach a contradiction since this time we would need to concentrate ϕ_2 increasing its L^2-norm and what can happen (and must happen) is that doing so $\bar{\phi_2} > \phi_1$. Finally we observe that we can apply exactly the same proof if $R_2 = \infty$, thus getting that $\nu(R) \geq \mathcal{V}$. □

We are now ready to show the convergence of ϕ_R.

Theorem 5.2. *The sequence ϕ_R converges to Φ in $L^2(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$ and we also have pointwise convergence.*

Proof. First of all we observe that the $\{\phi_R\}$ is bounded $H^1(\mathbb{R}^3)$-norm since $||\phi_R|| = \lambda$ and, by Lemma 3.2, $T(\phi) \leq 2(E(\lambda, R) + 2C/K\lambda^6) \leq K_1\lambda^6$ (at least for R big enough to have $E(\lambda, R) < 0$ and we know that for any λ such R exists by Lemma 3.1). This implies that, up to taking a subsequence, ϕ_R converges weakly in $L^2(\mathbb{R}^3)$ and in $H^1(\mathbb{R}^3)$ to some ϕ_∞ (thus $||\phi_\infty||_2 \leq \lambda$). Moreover (using the uniform bound on the L^2 and L^6 norm), we can find a radial decreasing function f, of order $r^{-3/2}$ as $r \to \infty$ and of order $r^{-1/2}$ as $r \to 0$, such that $\phi_R \leq f$ for any R. Using the arguments from [2] already cited in the proof of the existence of minimizers, we can then recover pointwise convergence, still up to taking subsequences, of ϕ_R to ϕ_∞. In particular this implies that ϕ_∞ is radial decreasing and nonnegative. We now make use of the previous Proposition 5.1, to have that $\nu(R)$ converges to some $\nu(\infty) \geq \mathcal{V}$ by monotonicity. This implies, using weak convergence and dominated convergence (with the domination given by f) that ϕ_∞ must satisfy, at least in a weak sense:

$$(-\Delta + U_{\phi_\infty})\phi_\infty = \nu(\infty)\phi_\infty.$$

If $\nu(\infty) > \mathcal{V}$ then, by the usual argument, we consider a rescaling which satisfies the equation with eigenvalue $\mathcal{V}$. This rescaled function will be obtained by flattening ϕ_∞ and reducing its L^2-norm and, by uniqueness, will then be equal to Φ. This implies $||\phi_\infty||_2 > \lambda$, which is a contradiction since we already know that by weak convergence $||\phi_\infty||_2 \leq \lambda$. Then it must be that $\nu(\infty) = \mathcal{V}$ and thus, by uniqueness, $\phi_\infty = \Phi$. This also implies $||\phi_\infty||_2 = \lambda$ which, together with the weak convergence, implies strong convergence in L^2. Now we consider $I(\phi_R) = \int |\phi_R(x)|^2/|x| dx$ and we observe that, by dominated convergence, $I(\phi_R) \to I(\phi_\infty)$. On the other hand, we recall that $\nu(\phi) = e(\phi) + 2I(\phi)$ and we have shown convergence of $\nu(\phi_R)$ and of $I(\phi_R)$ which together then imply $e(\phi_R) \to e(\Phi)$. To conclude we observe that $e(\phi_R) = \frac{1}{\lambda^2}(\mathcal{E}(\phi_R) - W(\phi_R)) = \frac{1}{\lambda^2}(T(\phi_R) - 2W(\phi_R))$. Since we have also $W(\phi_R) \to W(\Phi)$ by dominated convergence we can deduce $T(\phi_R) \to T(\Phi)$ which implies $||\phi_R||_{H^1(\mathbb{R}^3)} \to ||\Phi||_{H^1(\mathbb{R}^3)}$, which, together with $H^1(\mathbb{R}^3)$-weak convergence, implies $H^1(\mathbb{R}^3)$ strong convergence and completes the proof. We highlight that we have shown convergence up to subsequences, but, by the uniqueness of the limit point, this also implies convergence for the whole sequence. □

Corollary 5.3. *As a corollary, we can also show $E(\lambda, R) \to E(\lambda)$*

Proof. We just observe that $E(\lambda, R) = \mathcal{E}(\phi_R) = T(\phi_R) - W(\phi_R)$ and by what has been shown we have immediately that $\mathcal{E}(\phi_R) \to \mathcal{E}(\Phi) = E(\lambda)$. □

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