# A short trip to random matrices

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**Abstract.** In this text I give a brief summary of my doings during the stay at IST Austria and and some notes about the research task I was given and answered partially.

## 1 My work in general

After the introductory talk my professor gave me about the theory of random matrices (RMT) and their applications and motivations in physics I started to learn the proof of the Wigner's theorem (WT), which can be considered as the starting point of the RMT and the moment method in particular. For that purpose I used the book [1]. It took me quite a lot of time because RMT and the WT itself connects many branches of mathematics (probability theory, analysis, linear algebra, combinatorics). Then I was faced with the related task [1]. When I got stuck with answering it, Zhigang, one of the postdocs in my group, found quite a recent paper [2] solving almost the same problem. Their approach looked ingenious and very insightful to me [1] but for the main part didn't request any additional knowledge. Thus I went through it and my task was modified to get a similar result for slightly more general assumptions. Unfortunately it turned out that I am unable to give the result in such explicit form as in [2] since one particular PDE doesn't have explicit solution.

## 2 Wigner's theorem and the asymptotic expansion of expected empirical (spectral) measure

### 2.1 Introduction

Let us consider independent complex random variables $W_{ij}$ for $i, j \in \mathbb{N}$ with zero mean and finite moments of all order. Then we define Wigner matrix as $X_{ij}^{(n)} := \frac{1}{\sqrt{n}} W_{ij}$ and $X_{ji}^{(n)} := \overline{X_{ij}^{(n)}}$ for $1 \le i \le j \le n$. Every such matrix has real eigenvalues since it is hermitian. On certain assumptions for the Wigner matrices the WT states that the empirical (random) measures $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i}$ converge weakly in probability to the measure given by the density $\sigma(x) = \frac{1}{2\pi}\sqrt{4-x^2}\chi_{[-2,2]}$, i.e. for any bounded continuous $f$ one has

$$\frac{1}{n}\sum_{i=1}^n f(\lambda_i) = \int_{\mathbb{R}} f(x)\, d\mu_n(x) \xrightarrow[prob.]{n\to\infty} \int_{\mathbb{R}} f(x)\sigma(x)\, dx.$$

Using suitable continuous estimates of $\chi_{(a,b)}$ one can then for example check that $\frac{1}{n}|\{i\,|\lambda_i \in (a,b)\}| \xrightarrow[prob.]{} \int_a^b \sigma(x)\, dx$.

[1] See next section

There are two common ways to get this result – the *moment method* and the *resolvent method*. The moment method uses the moments

$$m_k = \mathbb{E}\int_{\mathbb{R}} x^k \, d\mu_n(x) = \mathbb{E}\frac{1}{n}\sum_i \lambda_i^k = \mathbb{E}\frac{1}{n}\operatorname{Tr} X^k \text{ }^2$$

and their counting involve the combinatorial stuff. The resolvent method computes the Stieltjes transform (2.4 in [1]) of the limit of the empirical measures $\mu_n$.

### 2.2 Asymptotic expansion, paper [2] and the new assumption

Let us now concentrate on the moment method. The substantial part of WT proof via moment method is dealing with the sequence $m_k(n)$, in particular showing that $\lim_{n\to\infty} m_{2k+1} = 0$ and $\lim_{n\to\infty} m_{2k} = \operatorname{Cat}(k)$, where $\operatorname{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$ is the $n$-th Catalan number. Actually this can be considered as a weaker version of WT (instead of the convergence in probability). When one wants to know more about $m_k(n)$ the asymptotic expansion in $\frac{1}{n}$ can be done. First related question arose is about the magnitude of the subleading term of the expansion. In greater detail we ask about

$$\lim_{n\to\infty} n\left(\mathbb{E}\int_{\mathbb{R}} x^k \, d\mu_n(x) - \int_{\mathbb{R}} x^k \sigma(x)\, dx\right).$$

The paper [2] accomplished this task and even managed to obtain explicit formula for „correction measure“ to the semicircle distribution[3].

The moment method proof of WT requires only the first two moments of the entries of $X$ and the boundedness of all the moments (see 2.1 in [1]). For computing the subleading term the paper [2] uses $\mathbb{E}W_{ij}^2 = 0$ for $i \neq j$ (for $i = j$ we have $\mathbb{E}W_{ii}^2 = \mathbb{E}|W_{ii}|^2 = \sigma^2 = \mathbb{E}|W_{mn}|^2$ as for any „non–diagonal“ variable $W_{mn}$) and moreover $\mathbb{E}|W_{ij}|^4 = \alpha$ also for $i \neq j$. Then one may ask what happens in the complex case if we don't make $\mathbb{E}W_{ij}^2$ disappear but set for all $i < j$ $\mathbb{E}W_{ij}^2 = \theta$ and $\mathbb{E}W_{ji}^2 = \overline{\theta}$ for some $\theta \in \mathbb{C}$.

### 2.3 Contribution of the new case

We will follow the approach of [2]. They showed that the expectation of odd moments is zero, so let $k = 2l$. We want to compute

$$\lim_{n\to\infty} n\left(\mathbb{E}\frac{1}{n}\left(\operatorname{Tr} X^{2l}\right) - \operatorname{Cat}(l)\right),$$

more specifically just the terms of

$$\lim_{n\to\infty} \frac{1}{n^l}\sum_{i_1,\dots,i_n=1}^{n} \mathbb{E}\left(X_{i_1 i_2}\dots X_{i_{2l} i_1}\right)$$

corresponding to the closed paths on the graphs on $\{1,\dots,n\}$ having $l$ vertices, $l$ edges and exactly one cycle and the path visits each edge twice. Furthermore it visits exactly the edges of the cycle both times in the same direction. Unlike the first case discussed on page 6 of [2] the contribution of such a term depends on the number of edges of the cycle

[2]We will write just $X$ instead of $X^{(n)}$.

[3]Reading first the mentioned paper is highly recommended.

visited from the lower index to the higher one and those visited in the other direction. In the spirit of [2] we obtain the expression of desired contribution

$$D_{2l}(\theta) := \lim_{n\to\infty} \frac{1}{n^l} \sum_{p=3}^{l} \frac{1}{\sigma^{2p}} n(n-1)\dots(n-l+p+1)\binom{n-l+p+1}{p} \cdot$$

$$\cdot \sum_{\substack{l_1+\dots+l_p\\+r_1+\dots+r_p=l-p}} (2l_1+1)\prod_{i=1}^{p} \operatorname{Cat}(l_i)\operatorname{Cat}(r_i) \sum_{a=0}^{p} \theta^a \left(\overline{\theta}\right)^{p-a} p\cdot \mathcal{C}(p,a).$$

where $\mathcal{C}(p,a)$ denotes the number of permutations $\pi$ of $\{1,\dots,p\}$ such that $|\{i\in\{1,\dots,p\}\,|\pi(i)<\pi(i+1)\}| = a$ where we set $\pi(p+1)=\pi(1)=1$. Now we put $\theta=|\theta|e^{i\varphi}$ and evaluate the limit:

$$D_{2l}(\theta) = \sum_{p=3}^{l} \sum_{\substack{l_1+\dots+l_p\\+r_1+\dots+r_p=l-p}} (2l_1+1)\prod_{i=1}^{p} \operatorname{Cat}(l_i)\operatorname{Cat}(r_i)\left(\frac{|\theta|}{\sigma^2}\right)^p e^{-pi\varphi}\sum_{a=0}^{p} e^{2ai\varphi}\frac{\mathcal{C}(p,a)}{(p-1)!}.$$

The number $\mathcal{C}(p,a)$ is determined by $\mathcal{C}(p,0)=0$, $\mathcal{C}(3,1)=\mathcal{C}(3,2)=1$ and the recurrence

$$\mathcal{C}(p,a) = a\mathcal{C}(p-1,a) + (p-a)\mathcal{C}(p-1,a-1).$$

One can check the recurrence by erasing the number $p$ from any such permutation and reconstructing it back by placing $p$ either in one of $a$ „increasing“ gaps in any of $\mathcal{C}(p-1,a)$ permutations or in one of $(p-1)-(a-1)=p-a$ gaps in any of $\mathcal{C}(p-1,a-1)$ permutations.

## 2.4 Generating functions

Our current purpose is to find the generating function of $D_k(\theta)$. In order to get this we need to get rid of the factorial so let us define $\mathcal{F}(p,a) := \frac{\mathcal{C}(p,a)}{(p-1)!}$. It satisfies the recurrence

$$(p-1)F(p,a) = a\mathcal{F}(p-1,a) + (p-a)\mathcal{F}(p-1,a-1)$$

and the initial conditions $\mathcal{F}(p,0)=0$, $\mathcal{F}(3,1)=\mathcal{F}(3,2)=\frac{1}{2}$.

Let us denote the generating function of $\mathcal{F}(p,a)$ by $f(x,y)=\sum_{p=3}^{\infty}\sum_{a=0}^{p}\mathcal{F}(p,a)x^p y^a$ and the generating function of Catalan numbers by $T(x)=\sum_{k=0}^{\infty}\operatorname{Cat}(k)x^k$. Now we can write the generating function of $D_k(\theta)$ as

$$\sum_{l=3}^{\infty} D_{2l}(\theta)x^l = \sum_{l=3}^{\infty}\sum_{p=3}^{l} \sum_{\substack{l_1+\dots+l_p\\+r_1+\dots+r_p=l-p}} (2l_1+1)\prod_{i=1}^{p}\operatorname{Cat}(l_i)\operatorname{Cat}(r_i)x^l\left(\frac{|\theta|}{\sigma^2}\right)^p e^{-pi\varphi}\sum_{a=0}^{p} e^{2ai\varphi}\mathcal{F}(p,a) =$$

$$= \frac{1}{T(x)}\sum_{p=3}^{\infty}\sum_{a=0}^{p}\left(\frac{|\theta|}{\sigma^2}e^{-\varphi i}xT(x)^2\right)^p\left(2xT'(x)+T(x)\right)\left(e^{2\varphi i}\right)^a\mathcal{F}(p,a) =$$

$$= \left(1+2x\frac{T'(x)}{T(x)}\right) f\left(\frac{\overline{\theta}}{\sigma^2}xT(x)^2, e^{2\varphi i}\right) =: G(x).$$

We need $f$ to be defined for the inputs above. But since $|e^{2\varphi i}|=1$, $\sum_{a=0}^{p}\mathcal{F}(p,a)=1$ for all $p\geq 3$ and $\lim_{x\to 0}T(x)=1$, for any $\theta\in\mathbb{C}$ we can find sufficiently small neighbourhood of $0\in\mathbb{C}$ where $G(x)$ converges absolutely.

It is well known that $T(x) = \frac{1-\sqrt{1-4x}}{2x}$ so it remains to find a formula for $f$. Then $G(x)$ would be a new term in the last section of [2] representing our slight generalization of assumptions. By plugging the reccurence for $\mathcal{F}(p,a)$ in the definition of $f(x,y)$ we get the equation

$$f(x,y) + x^3y + x^3y^2 + \frac{\partial f}{\partial x}(x,y)\left(x^2y - x\right) + \frac{\partial f}{\partial y}(x,y)\left(xy - xy^2\right) = 0.$$

Then we know $f(0,y) = f(x,0) = 0$. It is a first order PDE for which the characteristic method is used. Projections on the $xy$ plane of the characteristic are solutions of this system of ODEs:

$$\begin{aligned} x' &= x^2y - x \\ y' &= xy - xy^2 \end{aligned}$$

and are given by $y_r(x) = \frac{\log(x)+r}{x-1}$ for any $r \in \mathbb{R}$. They do not cross each other and cover some small neighbourhood of $0 \in \mathbb{R}^2$. Hence there is a unique solution of our PDE in some small neighbourhood of zero – our $f(x,y)$. But if this solution could be written explicitly then the inverse function to the $y_r$ would also have to be explicit which is not. This is thus the best result we can obtain via our approach.

## Reference


[1] Anderson, G.W., Guionnet, A., Zeitouni, O. (2010), *An Introduction to Random Matrices*, Cambridge University Press

[2] Enriquez N., Ménard L. (2015), *Asymptotic expansion of the expected spectral measure of Wigner matrices*, arXiv: 1506.03002v1 [math.PR]