The Local Semicircle Law for Random Matrices with a Fourfold Symmetry

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Notation and Assumptions

Let $H = (h_{xy}^{(N)})_{x,y \in \mathbb{Z}/N\mathbb{Z}}$ be an $N \times N$ random matrix with centered ($\mathbb{E} h_{xy} = 0$) independent real or complex entries up to the fourfold symmetry

$$h_{xy} = \bar{h}_{yx} = h_{-y,-x} = \bar{h}_{-x,-y}$$

for all $x, y \in \mathbb{Z}/N\mathbb{Z}$ and $N \in \mathbb{N}$.

Empirical spectral measure $\mu_N := \dfrac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i^{(N)}}$.

Wigner's semicircle law $\rho(x) := \frac{1}{2\pi} \sqrt{(4 - x^2)_+}$.
Stieltjes transform of probability measure μ on $\mathbb{R}$:

$$S(\mu) \colon \mathbb{C} \backslash \mathbb{R} \to \mathbb{C}, S(\mu)(z) := \int_{\mathbb{R}} (x - z)^{-1} \mathrm{d}\mu$$

Stieltjes transform of ρ: $m(z) := S(\rho)(z)$.
Resolvent of H: $G(z) := (H - z)^{-1}$ with entries $G_{ij}(z)$.

Stieltjes transform m_N of the empirical spectral measure $m_N(z) := S(\mu_N)(z) = N^{-1} \operatorname{tr} G(z)$.
Notation:

$$s_{xy} := \mathbb{E}|h_{xy}|^2, \qquad M := \left(\max_{x,y} s_{xy} \right)^{-1}, \qquad \zeta_{xy} := s_{xy}^{-1/2} h_{xy}.$$

Assumptions

- For all $x \in \mathbb{Z}/N\mathbb{Z}$ we assume
$$\sum_y s_{xy} = 1.$$
- There is a constant $\delta > 0$ such that
$$N^{\delta} \leq M \leq N.$$
- There are constants μ_p such that
$$\mathbb{E}|\zeta_{xy}|^p \leq \mu_p$$
for all $x, y \in \mathbb{Z}/N\mathbb{Z}$ and $N, p \in \mathbb{N}$.

Definition (Stochastic Domination)**.** *Let $X = (X^{(N)}(u); u \in U^{(N)}, N \in \mathbb{N})$ and $Y = (Y^{(N)}(u); u \in U^{(N)}, N \in \mathbb{N})$ be two families of nonnegative random variables for a possibly N-dependent parameter set $U^{(N)}$. We say that X is* stochastically dominated *by Y, uniformly in u, if for all $\varepsilon > 0$ and $D > 0$ there is a $N_0(\varepsilon, D) \in \mathbb{N}$ such that*

$$\sup_{u \in U^{(N)}} \mathbb{P}\left[X^{(N)}(u) > N^{\varepsilon} Y^{(N)}(u) \right] \leq N^{-D}$$

for all $N \geq N_0$. In this case, we use the notation $X \prec Y$.

Main Result

Theorem (Local Semicircle Law)**.** *If H fulfills the previous assumptions then*

$$|G_{ij}(z) - \delta_{ij} m(z)| \prec \sqrt{\frac{\operatorname{Im} m(z)}{M\eta}} + \frac{1}{M\eta}$$

uniformly in i, j and $z \in \mathbf{S}$, as well as

$$|m_N(z) - m(z)| \prec \frac{1}{M\eta}$$

uniformly in $z \in \mathbf{S}$ ($\eta = \operatorname{Im} z$).

Notation:

$$\Lambda(z) := \max_{x,y} |G_{xy}(z) - \delta_{xy} m(z)|.$$

Self-consistent Equations:

$$-\sum_a s_{xa} v_a + \mathcal{E}_x = \frac{1}{v_x + m} - \frac{1}{m}, \tag{1}$$

$$G_{x,-x} = m^2 \sum_{a \neq -a} (\mathbb{E} h_{xa}^2) G_{a,-a} + \mathcal{F}_x, \tag{2}$$

with $v_a := G_{aa} - m$ and error terms $\mathcal{E}_x$, $\mathcal{F}_x$.

Definition. *For a subset $\mathbb{T} \subset \mathbb{Z}/N\mathbb{Z}$ we define the minor $H^{(\mathbb{T})}$ through*

$$(H^{(\mathbb{T})})_{ij} := \mathbf{1}(i \notin \mathbb{T}) \mathbf{1}(j \notin \mathbb{T}) h_{ij}$$

and its resolvent or Green function

$$G^{(\mathbb{T})} := (H^{(\mathbb{T})} - z)^{-1}$$

with entries $G_{ij}^{(\mathbb{T})}(z)$.
For summations we use the convention

$$\sum_i^{(\mathbb{T})} := \sum_{i; i \notin \mathbb{T}}.$$

Notation:

$$\mathbb{F}_a X := X - \mathbb{E}[X | H^{(a,-a)}],$$
$$\Gamma_S := \|(1 - m^2 S)^{-1}\|_{\ell^\infty \to \ell^\infty})$$

with $S = (s_{xy})_{x,y}$.

References

[1] L. Erdős, A. Knowles, H.-T. Yau, J. Yin: *The local semicircle law for a general class of random matrices. Electron. J. Probab.*, **18**(2013), no. 59, 1–58.