Enumerative invariants of Higgs moduli spaces IV

joint work with András Szenes

Tamás Hausel

IST Austria & EPF Lausanne
http://hausel.ist.ac.at

Perspectives in Geometry and Topology
University of Texas, Austin
February 2017
Lecture 1: Cohomology of Higgs moduli via $\mathbb{C}^\times$ action
Lecture 2: Cohomology of character variety via arithmetic & $P = W$
Lecture 3: Cohomology of quiver varieties and Kac’s conjectures
Lecture 4: Equivariant Verlinde algebra for Higgs bundles
Recollection on Frobenius algebras

- $\mathbb{K}$ field of characteristic 0
- **Frobenius algebra**: finite dimensional commutative unital $\mathbb{K}$-algebra $F$ with non-degenerate pairing $\langle , \rangle$, which is symmetric $\langle a, bc \rangle = \langle ab, c \rangle$
  - e.g. $\mathbb{K}R(G)$ for finite group $G$ or $\mathbb{K}[G]^G$ with convolution
- $1+1D$ TQFT $\iff Z(S^1) = F$ with pairing $\langle , \rangle$
- $(a_i)$ basis of orthogonal idempotents then
  \[
  Z(\Sigma_g) = \sum_i \langle a_i, 1 \rangle^{1-g} \text{ Verlinde formula}
  \]
  - e.g. when $F = \mathbb{K}[G]^G$ we get $(\chi | G|^{-1/2})_{\chi \in \hat{G}}$ orthogonal idempotents and
  \[
  Z(\Sigma_g) = \sum_{\chi \in \hat{G}} \left( \frac{|G|^2}{\chi(1)^2} \right)^{g-1} = \frac{1}{|G|} |\text{Hom}(\pi_1(\Sigma_g), G)|
  \]
- $1+1D$ Chern-Simons theory with finite gauge group of (Freed–Quinn, 1993)
- another example $SU(n)$ Chern-Simons theory
  $\sim$ Verlinde algebra discussed next
Verlinde formulae

- \( C \) smooth complex projective curve of genus \( g \)
- fix rank \( n \in \mathbb{Z}_{>0} \), degree \( d \in \mathbb{Z} \) and level \( k \in \mathbb{Z}_{>0} \)
- \( \mathcal{N}_n^d \) moduli space of semi-stable rank \( n \) fixed degree \( d \) vector bundles on \( C \); projective and smooth when \((d, n) = 1\)
- \( L \in \text{Pic}(\mathcal{N}_n^d) \cong \mathbb{Z} \) ample generator of Picard group
- Verlinde formula (1988) for \( \dim H^0(\mathcal{N}_n^d; L^k) = \chi(\mathcal{N}_n^d, L^k) \)
- e.g. for \( n = 2 \) \( d = 1 \)

\[
\dim H^0(\mathcal{N}_2^1, L^k) = \sum_{j=1}^{2k+1} (-1)^{j+1} \left( \frac{k+1}{\sin^2\left(\frac{j\pi}{2k+2}\right)} \right)^{g-1} = \\
\frac{1}{2} \text{Res}_{z=1} \left( \frac{(4k+4)^g}{(z^{k+1}-z^{-(k+1)})(1-1/z)(1-z)^{g-1}} \right) \frac{dz}{z}
\]

- proved for
  - \( k = 1 \) by (Beauville–Narasimhan–Ramanan 1988)
  - \( n = 2 \) by (Szenes, Bertram–Szenes 1993)
  - : ...
  - in all generality by (Teleman–Woodward, 2009)
Verlinde algebra

- Verlinde formula = partition function of a 1 + 1D TQFT
- 1 + 1D TQFT determined by a Frobenius algebra
  i.e. finite dimensional comm. \(\mathbb{C}\)-algebra + symmetric pairing
- \(R := R(SU_n) \cong\) character ring of \(SU_n\)
- \(R \cong R(T_n)^{S_n} \cong (\mathbb{Z}[z_1, \ldots, z_n]/(z_1 \cdots z_n - 1))^{S_n}\)
- irrep \(\chi_\lambda \in \text{Irr}(SU_n) \leftrightarrow s_\lambda \in R\) Schur function
  \(\lambda = (\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0) \in \mathbb{Z}^{n-1}\)
- \(\text{Ver}^k_n := \mathbb{C} \otimes_{\mathbb{Z}} R/(s_{(k+1)}, s_{(k+2)}, \ldots, s_{(k+n-1)})\) has basis
  \(\{s_\lambda\}_{\lambda_1 \leq k}\)
- declaring \(\langle s_\lambda, s_\eta^\dagger \rangle = \delta_{\lambda\eta} \sim \) non-degenerate pairing

**Theorem (Goodman-Wenzl 1990; Gepner 1991; Witten 1993)**

\((\text{Ver}^k_n, \langle, \rangle)\) is a Frobenius algebra (i.e. \(\langle a, bc \rangle = \langle ab, c \rangle\)).
\(\cong\) Verlinde algebra, with partition function giving Verlinde formulae

- geometrization: Borel-Weil-Bott theory \(\sim\)
  \(\chi_{T_n}(\mathcal{F}; L_\lambda) = s_\lambda \in R(T_n)^{S_n}\) on flag variety \(\mathcal{F} := SU_n/T_n\)
3:30 pm   Thursday, September 18, 2003

Geometry Seminar: **Integrals on the moduli of Higgs bundles**
by **Tamas Hausel** (UT - Math) in RLM 9.166

First I explain how one can integrate in the circle equivariant cohomology of the non-compact moduli space of Higgs bundles on a Riemann surface. Then I explain why it is presently impossible to calculate these integrals. Then I mention that the physicists Moore-Nekrasov-Shatashvili do have formulas, surprisingly related to certain Bethe Ansatz equations, for the equivariant volumes of these spaces, a result which is obtained using Feynman integral methods. Then I show some calculational results with Andras Szenes which makes us conjecture that a miraculous agreement in the contribution to our integrals will give a general formula for this incalculable calculation, and also gives a new, fairly elementary, approach to Witten's intersection numbers on the moduli space of stable bundles. We will calculate integrals on the Higgs moduli space related to equivariant volumes, y-genuses and the Verlinde formulas, and finish with a review of possible applications of these ideas.

Submitted by **Vittoria Esile**
Equivariant Verlinde formulae

- \( \mathcal{M}_n^d \supset T^* \mathcal{N}_n^d \) moduli ss rank \( n \) fixed degree \( d \) Higgs bundles

i.e. Higgs bundles of the form \( (E, \phi) \) where \( \text{rank}(E) = n \),
\( \text{det}(E) = \Lambda \), \( \text{deg}(\Lambda) = d \) and \( \phi \in \text{End}_0(E) \otimes K \)

- \( \mathbb{T} := \mathbb{C}^\times \) acts on \( \mathcal{M}_n^d \) by scaling Higgs field

- \( L \in \text{Pic}(\mathcal{M}_n^d) \) ample generator with \( \mathbb{T} \) action trivial on \( L^k |_{\mathcal{N}_n^d} \)

- \( \mathbb{T} \) acts on \( H^0(\mathcal{M}_n^d, L^k) \) with weights \( \geq 0 \)

- \( \text{grdim}(H^0(\mathcal{M}_n^d, L^k)) = \sum_{i=0}^{\infty} \text{dim}(H^0(\mathcal{M}_n^d, L^k)^i) t^i \in \mathbb{Z}[[t]] \)

- (Paradan 2011) \( \sim \)

\[ \chi_\mathbb{T}(\mathcal{M}_n^d, L^k) = \sum_{F_i} \int_{F_i} \text{ch}_\mathbb{T}(L^k|_{F_i} \otimes \text{Sym}_t N^* F_i) \text{Todd}(TF_i) \]

- \( F_i \subset (\mathcal{M}_n^d)^\mathbb{T} \) fixed point components

- (Hausel–Szenes, 2003) direct computation \( \sim \)

\[ \chi_\mathbb{T}(\mathcal{M}_2^1, L^k) = \sum_{a=1,1/t,1/t} \text{Res}_{z=a} \left[ \frac{2^{2g-1} k+1 + \frac{zt}{1-zt} + \frac{t/z}{1-t/z}}{(1-t)^{g-1}} \right] g \]

- (Hausel–Szenes, 2003) conjecture for higher \( n \)

recently (Halpern-Leistner 2016) and (Andersen–Gukov–Pei 2016) gave formulas for \( \chi_\mathbb{T}(\mathcal{M}_G, L^k) \) for general \( G \) building on the work of (Teleman–Woodward 2009)
by physics arguments (Gukov–Pei 2015) \( \chi_T(\mathcal{M}_n^d, L^k) \) arises from a 1 + 1D TQFT, i.e. a Frobenius algebra, dubbed *equivariant Verlinde algebra*

we construct this algebra explicitly:

*step 1:* find \( t \)-deformation \( R_t(SU_n) \) of \( R(SU_n) \) and of basis \( s_\lambda \)

*step 2:* find deformed ideal \( I_t \) in the deformation

\[ QVer_n^k := \mathbb{C}(t) \otimes_{\mathbb{Z}(t)} R_t(SU_n)/I_t \]

*step 3:* define pairing \( \langle , \rangle_t \) and show it yields symmetric algebra \( (QVer_n^k, \langle , \rangle_t) \)

*step 4:* check partition function giving \( \chi_T(\mathcal{M}_n^d, L^k) \) and compare with properties described in (Gukov–Pei, 2015), (Gukov–Pei–Yan–Ye 2016), (Andersen–Gukov–Pei, 2016)
step 1. Definition of $R_t(SU_n)$ and $\chi^t_\lambda$

- recall geometrisation of $R(SU_n)$:
  $$\chi_{T_n}(\mathcal{F}; L_\lambda) = \chi_\lambda \in R(T_n)^{S_n}$$
on flag variety $\mathcal{F} := SU_n/T_n$

- consider $T^*\mathcal{F}$ and $L_\lambda := \pi^*L_\lambda \in \text{Pic}_T(T^*\mathcal{F})$

- $\chi^t_\lambda := \chi_{T_n \times T}(T^*\mathcal{F}, L_\lambda) \in R(T_n \times \hat{T})^{S_n} \cong R(SU_n)[[t]] =: R_t(SU_n)$

- $\chi^t_\lambda$ computed by (Gupta/Brylinski 1987):
  $$\chi^t_\lambda = E_\lambda \in R(T_n)^{S_n}[[t]]$$
can be obtained
  $$E_\lambda = t_\lambda(t)P_\lambda/\psi_t$$
  $$P_\lambda = \sum_{w \in S_n} (-1)^{\sigma(w)}z^{w(\lambda)}w(\Delta_t) / \Delta_1 t_\lambda(t) \in R(SU_n)[t]$$
  Hall-Littlewood
  $$\Delta_t = z^\rho \prod_{\alpha \in \Phi^-} (1 - tz^\alpha); \psi_t = \prod_{\alpha \in \Phi}(1 - tz^\alpha)$$
  $$t_\lambda(t) = \sum_{w \in S^\lambda_n} t^{l(w)}$$

Theorem (Gupta 1987)

$$\langle P_\lambda, E_{\eta^\dagger} \rangle = \frac{1}{n!} \text{Res}_{z=0} \frac{E_\lambda \psi_t}{t_\lambda(t)} E_{\eta^\dagger} \psi_1 \frac{dz}{z} = \delta_{\lambda \eta}$$
step 2. Definition of \( QVer_n^k \)

- for \( \alpha \in \Phi \) define
  \[
  b_\alpha := z^{(k+n)\alpha} \prod_{\beta \in \Phi} (1 - tz^\beta)^{-\langle \alpha, \beta \rangle} \in R(T_n)(t)
  \]
- \( \sim \) non-symmetric Bethe-Ansatz equation \( b_\alpha = 1 \)
- we have \( b_{\alpha+\beta} = b_\alpha b_\beta \) and \( b_{w\alpha} = w(b_\alpha) \)
- thus \( I'_t := (1 - b_{\alpha_1}, \ldots, 1 - b_{\alpha_{n-1}}) = (1 - b_\alpha)_{\alpha \in \Phi} \triangleleft R(T_n)(t) \)
- then \( \text{Spec}(\mathbb{F}R(T_n)/I'_t) \subset T_n(\mathbb{F}) \) with \( \mathbb{F} := \mathbb{C}(t) \)
- i.e. the solutions of the Bethe-Ansatz equations \( b_{\alpha_1} = 1, \ldots, b_{\alpha_{n-1}} = 1 \) in \( T_n(\mathbb{F}) \) are invariant under \( S_n \)
- \( \text{Spec}_n^k := \text{Spec}(\mathbb{F}R(T_n)/I'_t) \setminus \text{Spec}(\mathbb{F}R(T_n)/(1 - z^\alpha)_{\alpha \in \Phi}) \)
- for \( \lambda = (k + 1 = \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0) \) form
  \[
  B_\lambda = \frac{\sum_{w \in S_n} (-1)^{\sigma(w)} z^w(\lambda) (1 - tz^w(-\theta)) w(\Delta_t)}{\Delta_1} \in R(T_n)^{S_n}[t]
  \]
  symmetric Bethe-Ansatz polynomial
- then \( B_\lambda \Delta_1 \in I'_t \)
- \( \lambda_m := (k + 1, 1, \ldots, 1, 0, \ldots, 0) = k \omega_1 + \omega_m \)
- \( I_t := (B_{\lambda_1}, \ldots, B_{\lambda_{n-1}}) \triangleleft R(T_n)^{S_n}[t] \)
- \( QVer_n^k := \mathbb{F}R(T_n)^{S_n}[t]/I_t \)
- then \( \text{Spec}_n^k/S_n \subset \text{Spec}(QVer_n^k) \subset T_n(\mathbb{F})/S_n \)
step 3. \((\text{QVer}_n^k, \langle \cdot, \cdot \rangle_t)\) is a Frobenius algebra

\[
\langle E_\lambda, E_{\eta^\dagger} \rangle_t := \delta_{\lambda \eta} \tilde{t}_\lambda(t)(1 - t)^{n-1}, \quad \tilde{t}_\lambda(t) = \sum_{w \in S_{t\tilde{\crs}_n}(\lambda)} t^{l(w)}
\]

Theorem (Hausel–Szenes 2016)

\((E_\lambda)_{\lambda_1 \leq k}\) is a basis of \(\text{QVer}_n^k\), and \(\langle a, bc \rangle_t = \langle ab, c \rangle_t\)

proof: assume \(l_1(\lambda), l_1(\eta) \leq k\) and denote

\[
r = \tau_{k\omega_1} \circ W(1,2,...,n) : \mathbb{Z}^{n-1} \to \mathbb{Z}^{n-1}\text{ then prove:}
\]

\[
\langle E_\lambda, E_{\eta^\dagger} \rangle_t / (1 - t)^{n-1}
\]

\[
= \frac{1}{n!} \sum_{a \in \text{Spec}_k^n} \frac{E_\lambda(a) E_{\eta^\dagger}(a) \psi_t(a) \psi_1(a)}{\text{Jac}(\log(b_{\alpha_1}),...\log(b_{\alpha_{n-1}}))(a)}
\]

\[
= \frac{(-1)^{n-1}}{n!} \sum_{a \in \text{Spec}_k^n} \text{Res}_{z=a} \frac{E_\lambda E_{\eta^\dagger} \psi_t \psi_1}{(1-b_{\alpha_1})...(1-b_{\alpha_{n-1}})} \frac{dz}{z}
\]

\[
= \frac{(-1)^{n-1}}{n!} \sum_{a \in \text{Spec}_k^n} \text{Res}_{z=a} \frac{E_r(\lambda) E_{r(\eta)^\dagger} \psi_t \psi_1}{(1-b_{\alpha_1})...(1-b_{\alpha_{n-1}})} \frac{dz}{z}
\]

observe for some \(r^m\) no residue at infinity \(\Rightarrow\)

\[
= \frac{1}{n!} \text{Res}_{z=0} \frac{E_r^m(\lambda) E_{r^m(\eta)^\dagger} \psi_t \psi_1}{(1-b_{\alpha_1})...(1-b_{\alpha_{n-1}})} \frac{dz}{z} \overset{\text{Gupta}}{=} \delta_{r^m(\lambda)} r^m(\eta) t_{r^m(\lambda)}(t) = \delta_{\lambda \eta} \tilde{t}_\lambda(t)
\]

\[
\Rightarrow |(\lambda)_{\lambda_1 \leq k}| = \binom{n+k-1}{n-1} \leq \dim R_t/I_t \overset{\text{Bezout}}{\leq} \binom{n+k-1}{n-1} \quad \blacksquare
\]
step 4. Partition function and other checks

- ⇒ rotation on $QVer_n^k$ corresponds to multiplying by $\tilde{P}_{k\omega_1} = P_{k\omega_1} t_{k\omega_1}(t)/\tilde{t}_{k\omega_1}(t)$, and $\tilde{P}_{k\omega_1}^d = \tilde{P}_k(\omega_1+\cdots+\omega_d) = \tilde{P}_{k\omega_1 d}$

- $Tr := QVer_n^k \to \mathbb{F}$ by $Tr(E) := \frac{1}{n!} \sum_{a \in \text{Spec}_n^k} E(a) \psi_t(a) \psi_1(a) (1-t)^{n-1}$

then $\Rightarrow \langle E_1, E_2 \rangle_t = Tr(E_1 E_2)$

- we have $Z_n^k(C_d^g) =$

$$= \frac{1}{n!} \sum_{a \in \text{Spec}_n^k} \tilde{P}_{k\omega_1 d}(a) \left( \frac{\text{Jac}(\log(b_{\alpha_1}),\ldots,\log(b_{\alpha_{n-1}}))(a)}{\psi_t(a) \psi_1(a) (1-t)^{n-1}} \right)^g$$

$$= \frac{(-1)^{n-1}}{n!} \sum_{a \in \text{Spec}_n^k} \text{Res}_{z=a} \tilde{P}_{k\omega_1 d} \text{Jac}(\log(b_{\alpha_1}),\ldots,\log(b_{\alpha_{n-1}}))^g \frac{dz}{z}$$

" = " $\frac{1}{n!} \sum_{a \in V(\psi_t \psi_1)} \text{Res}_{z=a} \tilde{P}_{k\omega_1 d} \text{Jac}(\log(b_{\alpha_1}),\ldots,\log(b_{\alpha_{n-1}}))^g \frac{dz}{z}$

- when $n = 2$, $d = 1$ this agrees with (Hausel–Szenes 2003)

$$\sum_{a=1,t,1/t} \text{Res}_{z=a} \frac{2^{g-1} (1-t)^{g-1}}{(1-t)^{g-1}} \left[ k+1+ \frac{t/z}{1-t/z} + \frac{tz}{1-tz} \right]^g \frac{dz}{z}$$

- $QVer_2^k$ and $QVer_3^k$ matches (Gukov–Pei–etal, 2015, 2016)
Geometric information in $\chi_T(\mathcal{M}_n^d, L^k)$

- $\chi : \mathcal{M}_n^d \to A \cong \bigoplus_{i=2}^n H^0(C; K_C^i)$
  
  Hitchin map

- Hitchin map for $GL_1 : (L, \phi) \in \mathcal{M}_{Dol}^{d;1} \cong J^d(C) \times H^0(C; K_C)$

  $\chi$ is projection to second factor:

  $\chi : J^d(C) \times H^0(C; K_C) \to H^0(C; K_C)$

- In general $\chi$ is a completely integrable Hamiltonian system

  Generic fibre $\chi^{-1}(a)$ is isomorphic with an Abelian variety

  E.g. $\chi^{-1}(a) \cong Prym(C_a/C)$ for $SL_n$

- $V := \chi_*(L^k)$ is a $\mathbb{T}$-equivariant vector bundle on $A \sim \to$

  $V = V_0 \otimes O_A$ trivial $\mathbb{T}$-bundle (Masuda–Moser-Jauslin-Petrie 1995)

- $\text{char}(V_0) = \chi_T(C_n^d; L^k) \in \mathbb{N}[t] \sim \to \text{char}(V_0)(0) = \chi(\mathcal{N}_n^d, L^k)$

  $V_a = \chi_T(\chi^{-1}(a), L^k)$ for generic $a \in A \sim \to$

  $\text{rank}(V) = n^g(nk)(n^2-1)(g-1)$

- $\chi_T(\mathcal{M}_n^d, L^k) = \frac{\text{char}(V_0)}{\prod_{i=2}^n (1-t^i)^{(2i-1)(g-1)}}$

- E.g. $g = 2 \sim \to \chi_T(\mathcal{M}_2^1; L^2) = \frac{49t^3+113t^2+75t+19}{(1-t^2)^3}$
Problems

- We can analogously define \((QVer_n^k, \langle , \rangle_t)\) for any compact simply-connected semisimple Lie group \(G\). Is it a Frobenius algebra?
- We understand how to read off \(\chi(M_{\mu}, L^k)\) for parabolic Higgs bundles \(M_{\mu}\) from \(QVer_n^k\). How to include irregular singularities? (Fredrickson-Pei-Yan-Ye 2017)
- Hall–Littlewood polynomials deform to Macdonald polynomials. Is there a corresponding further deformation of \(QVer_n^k\)? What does it compute?
- What TQFT computes equivariant Hirzebruch \(y\)-genera of \(M_n^d\)?
- Can we enhance our 1 + 1D TQFT to a 2 + 1D TQFT deforming the Jones–Witten TQFT?
- Is there a representation theory of deformations of affine Kac-Moody algebras or Hecke algebras/quantum groups at root of unity behind \(QVer_n^k\)?
- Can we relate \(\chi_T(M_n^d, L^k)\) along the Hitchin map to the abelianization program of (Atiyah–Hitchin 1987)?